THE LYAPUNOV SPECTRUM OF BLOCKDIAGONAL SEMILINEAR CONTROL SYSTEMS

Stefan Grünvogel
Institut für Mathematik, Universität Augsburg,
Universitätsstr. 14, 86135 Augsburg, Germany,
E-Mail: birdland@malaga.math.Uni-Augsburg.de
Phone: (++49)-(0)821/598-2244
Fax: (++49)-(0)821/598-2339
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Abstract: For semilinear control systems the Lyapunov spectrum is approximated via the Floquet and the Morse spectrum. If the semilinear control system possesses singular subspaces, which correspond to blockdiagonality of the system, it can be shown that the analysis of these two spectra can be reduced to these lower-dimensional systems. The Floquet spectrum of control sets with nonvoid interior is contained in the Floquet spectra on the restricted control systems. The Morse spectrum is contained in the Morse spectra of the restricted systems. In particular, all Lyapunov exponents are attained on the singular subspaces.

Keywords: semilinear systems, blockdiagonal, singular subspace, control flow, Lyapunov spectrum, Floquet spectrum, Morse Spectrum, projection, control set

1. Introduction

We are interested in the Lyapunov spectrum of semilinear control systems

$$\begin{aligned} &\dot{x}(t) = A(u(t))x(t), \qquad t \in \mathbb{R}, \qquad x(0) = x_0 \in \mathbb{R}^d \setminus \{0\}, \\ &u(\cdot) \in \mathcal{U} := \{u(\cdot) : \mathbb{R} \to \mathbb{R}^d : \text{measurable}, u(t) \in \mathrm{U} \text{ a.e.}\}, \end{aligned} \tag{1.1}$$

where $A : V \to \mathrm{Mat}(d, \mathbb{R})$ is continuous on an open set V, and possesses subspaces, which are invariant under every applied control. We denote such invariant subspaces as singular. We look at the case, where we can decompose the semilinear control system into certain subsystems by restriction. If $\mathbb{R}^d$ is decomposable into a direct sum of singular subspaces, this corresponds to a blockdiagonalization of $A(u)$ in a certain base. For instance this situations occurs, if our system possesses some state-space symmetries. This means, that $A(u)$ commutes with the action of some transformation group. The linear maps, which commute with a transformation group have some invariant subspaces in common, see e.g. [**?**] (Aston et al., 1994), Theorem 2.8. If one applies this to the right-hand side of (1.1), we get a decomposition of the semilinear control system into singular subspaces.

The Lyapunov spectrum of semilinear control systems has been investigated in detail by Colonius and Kliemann, see e.g. [**?**] (Colonius et al., 1996), and [**?**] (Colonius et al., 1996) for the control flow point of view. Our attempt to analyse the Lyapunov spectrum follows their way. The Lyapunov spectrum is approximated from 'within' by the Floquet spectrum, and an 'outer' approximation is made via the the Morse spectrum. Then the Lyapunov spectrum is sandwiched in between. In focus of this paper lies therefore the Floquet and the Morse spectrum of the semilinear control system. The blockdiagonal form of the semilinear control systems we investigate makes it possible to simplify the analysis of these two spectra. We will show, that these spectra are characterized by the reduced systems on the singular subspaces. This is an important fact for the numerical analysis of the Lyapunov spectrum. Instead of working with one high-dimensional system, the computational work can be distributed to several lower-dimensional control systems.

The outline of the paper is as follows. In the next section, we give a brief review of the control theoretic background we need. The exponential growth behavior of linear differential equations (or linear flows) can be studied via the associated (angular) system on the projective space $\mathbb{P}^{d-1}$. If the system is decomposable into singular subspaces, we get new control systems by restriction to these subspaces.

In the third section, the Floquet spectrum stands in the center of view. The problem is, that we can not directly apply the results of [?] (Colonius et al. 1996), because if the semilinear control system is decomposable into singular subspaces, the necessary non-degeneracy assumption is violated. Hence we introduce the concept of the projection of points in $\mathbb{P}^{d-1}$ on the projected singular subspaces. With help of this projection it is shown, that we can project control sets with nonvoid interior in $\mathbb{P}^{d-1}$ down to the projected singular subspaces. This enables us to characterize the Floquet spectrum of the main system by the Floquet spectra of the projected control system.

For analysing the Morse spectrum, we have to restrict ourselves to a subclass of semilinear control systems, the bilinear control systems. Here we can introduce the notion of control flow and use topological dynamics to analyse the spectrum. As with the Floquet spectrum, our attempt is to characterize the Morse spectrum with help of the restricted control systems. The result is, that we actually can restrict us to this subsystems, because on these subsystems all Morse exponents are attained. For numerical computations this means, that we only have to compute the Lyapunov spectrum of lower dimensional systems to get the whole Lyapunov spectrum.

In the fifth section, we give conditions under which the Floquet, the Lyapunov and the Morse spectra on the singular subspaces all coincide. This result will be used in the last section to compute the Lyapunov spectrum of four coupled oscillators.

2. Preliminaries

We consider the semilinear control system

$$\begin{aligned} &\dot{x}(t) = A(u(t))x(t), \qquad t \in \mathbb{R}, \qquad x(0) = x_0 \in \mathbb{R}^d \setminus \{0\}, \\ &u(\cdot) \in \mathcal{U} := \{u(\cdot) : \mathbb{R} \to \mathbb{R}^d : \text{measurable}, u(t) \in \mathrm{U} \text{ a.e.}\}, \end{aligned} \tag{2.1}$$

where $\mathrm{U} \subset \mathbb{R}^d, A : V \to \mathrm{Mat}(d, \mathbb{R})$ is continuous on an open set $V \supset \mathrm{U}$. Here $\mathrm{Mat}(d, \mathbb{R})$ denotes the space of real $d \times d$-matrices. We will assume, that (2.1) has for every $x \in \mathbb{R} \setminus \{0\}$ and every $u(\cdot) \in \mathcal{U}$, a unique trajectory $\varphi(\cdot, x, u(\cdot))$ on $\mathbb{R}$ with $\varphi(0, x, u(\cdot)) = x$. Sometimes, we will denote the semilinear control system as the *main* system.

We say, that a subspace $W \leq \mathbb{R}^d, W \neq \{0\}$ is *singular*, if

$$A(u)W \subseteq W \tag{2.2}$$

for all $u \in \mathrm{U}$. In this article, our main interest lies in the case, if we can decompose the space $\mathbb{R}^d$ into an direct sum of singular subspaces.

Definition 2.1. The state space $\mathbb{R}^d$ of the control system (2.1) is *decomposable* into singular subspaces $W_1, \ldots, W_K \neq \{0\}, K \in \mathbb{N} \setminus \{0\}$ if $\mathbb{R}^d = W_1 \oplus \ldots \oplus W_K$ and

$$A(u)W_i \subseteq W_i, \text{ for all } u \in \mathrm{U} \text{ and all } i \in \{1, \ldots, K\}.$$

If the state space of the control system (2.1) is decomposable into singular subspaces, this corresponds to a blockdiagonalization of $A(u)$ in a certain base of $\mathbb{R}^d$ for every $u \in \mathrm{U}$.

The restrictions of our semilinear control system (2.1) on these singular subspaces W_i are again semilinear control systems. These semilinear control systems are given by

$$\begin{array}{l} \dot{x}_i(t) = A(u(t))|_{W_i} x_i(t), \qquad t \in \mathbb{R}, \qquad x_i(0) = x_i^0 \in W_i \setminus \{0\}, \\ u(\cdot) \in \mathcal{U} := \{u(\cdot) : \mathbb{R} \to \mathbb{R}^d : \text{measurable}, u(t) \in \text{U a.e.}\}, \end{array} \tag{2.3}$$

on the singular subspaces W_i. For every $i \in \{1, \dots, K\}$ we denote by $\varphi_i(\cdot, x_i, u(\cdot))$ the associated trajectories, which are unique on $\mathbb{R}$ according to the assumptions on our main control system (2.1). Note, that for all $u(\cdot) \in \mathcal{U}$ and all $x_i \in W_i$ we have

$$\varphi(t, x_i, u(\cdot)) = \varphi_i(t, x_i, u(\cdot)), \text{ for all } t \in \mathbb{R}.$$

This motivates the term 'singular' subspaces. Similar to an equilibrium point, which is usually denoted as singular point, trajectories starting in the singular subspaces, remain in them.

We are interested in a characterization of the entire Lyapunov spectrum of (2.1), i.e. in all *Lyapunov exponents*

$$\lambda(u(\cdot), x) = \limsup_{t \to \infty} \|\varphi(t, x, u(\cdot))\|, \tag{2.4}$$

where $(u(\cdot), x) \in \mathcal{U} \times \mathbb{R}^d$. We define the Lyapunov spectrum of (1.1) as

$$\Sigma_{Ly} = \{\lambda(u(\cdot), x) : (u(\cdot), x) \in \mathcal{U} \times \mathbb{R}^d, x \neq 0\} \tag{2.5}$$

Because of linearity of (2.1), it holds that $\lambda(u(\cdot), x) = \lambda(u(\cdot), \alpha x)$ for all $\alpha \neq 0$. Thus the exponential growth behavior of linear differential equations can be studied via the associated (angular) system on the projective space $\mathbb{P}^{d-1}$. The *projected* control system is defined as

$$\begin{array}{l} \dot{p}(t) = h(p(t), u) = [A(u) - p^T(t) A(u) p(t) Id] p(t), \ t \in \mathbb{R}, \ p(0) = p_0 \in \mathbb{P}^{d-1}, \\ u(\cdot) \in \mathcal{U} := \{u(\cdot) : \mathbb{R} \to \mathbb{R}^d : \text{measurable}, u(t) \in \text{U a.e.}\}, \end{array} \tag{2.6}$$

If we have a state space decomposition of the semilinear control system (2.1) into a direct sum of K singular subspaces W_i, we get the following K control systems on the *projective singular* subspaces $\mathbb{P}(W_i)$:

$$\begin{array}{l} \dot{p}_i(t) = h(p_i(t), u) = [A(u)|_{W_i} - p^T(t)\ A(u)|_{W_i} p(t) Id_{W_i}] p(t), \\ t \in \mathbb{R}, \ p(0) = p_i^0 \in \mathbb{P}(W_i), \\ u(\cdot) \in \mathcal{U} := \{u(\cdot) : \mathbb{R} \to \mathbb{R}^d : \text{measurable}, u(t) \in \text{U a.e.}\}. \end{array} \tag{2.7}$$

Here $\mathbb{P}(\cdot)$ denotes the natural projection $\mathbb{P} : \mathbb{R}^d \setminus \{0\} \to \mathbb{P}^{d-1}$. For subspaces W_i we just write $\mathbb{P}(W_i)$ instead of $\mathbb{P}(W_i \setminus \{0\})$.

We denote the solutions of (2.6) by $\mathbb{P}\varphi(\cdot, p_0, u(\cdot))$ and by $\mathbb{P}\varphi_i(\cdot, p_i^0, u(\cdot))$ the solutions of the projected control systems on the projective singular subspaces $\mathbb{P}(W_i)$. Note, that for all $u(\cdot) \in \mathcal{U}$ and all $p_i \in \mathbb{P}(W_i)$ the following equality holds

$$\mathbb{P}\varphi(t, p_i, u(\cdot)) = \mathbb{P}\varphi_i(t, p_i, u(\cdot)), \qquad \forall t \in \mathbb{R}$$

Using piecewise constant controls, one can associate to the system (2.1) the systems group and semigroup, respectively:

$$\mathcal{G} := \left\{ \exp(t_n B_n) \dots \exp(t_1 B_1) : \begin{array}{l} t_j \in \mathbb{R}, B_j = A(u_j), u_j \in \mathrm{U}, \\ j = 1, \dots n, n \in \mathbb{N} \end{array} \right\} \subset \mathrm{GL}(d, \mathbb{R}) \tag{2.8}$$

$$\mathcal{S} := \left\{ \exp(t_n B_n) \dots \exp(t_1 B_1) : \begin{array}{l} t_j \geq 0, B_j = A(u_j), u_j \in \mathrm{U}, \\ j = 1, \dots n, n \in \mathbb{N} \end{array} \right\} \subset \mathrm{GL}(d, \mathbb{R}) \tag{2.9}$$

The sets $\mathcal{G}_t$ and $\mathcal{S}_t$ denote the group and semigroup elements *at time* t, i.e. with $\sum_{j=1}^{n} |t_j| = t$, and $\mathcal{G}_{\leq t}$ and $\mathcal{S}_{\leq t}$ denote group and semigroup elements with $\sum_{j=1}^{n} |t_j| \leq t$, Note that $\mathcal{G}$ and $\mathcal{S}$ act on a natural way on both $\mathbb{R}^d$ and $\mathbb{P}^{d-1}$ via $g(x) = g \cdot x$ and $g(p) = \mathbb{P}(g \cdot \mathbb{P}^{-1}(p))$. We do not distinguish in our notation between these two actions, because it is always clear from the context, which one is referred to.

The positive (and negative, respectively) orbit of an element $x \in \mathbb{P}^{d-1}$ up to time $T \geq 0$ is given by

$$\begin{aligned} \mathcal{O}^{+}_{\leq t}(x) &= \{ y \in \mathbb{P}^{d-1} : \text{there is } g \in \mathcal{S}_{\leq t} \text{ with } y = gx \} \\ \mathcal{O}^{-}_{\leq t}(x) &= \{ y \in \mathbb{P}^{d-1} : \text{there is } g \in \mathcal{S}_{\leq t} \text{ with } x = gy \} \end{aligned} \tag{2.10}$$

$\mathcal{O}^{+}(x)$ (and $\mathcal{O}^{-}(x)$) are the orbits $\mathcal{O}^{+}_{<\infty}(x)$ (and $\mathcal{O}^{-}_{<\infty}(x)$), i.e. with respect to the entire semigroup $\mathcal{S}$.

Definition 2.2. A nonempty set $D \subset \mathbb{P}^{d-1}$ is called a *control set* of the control system (2.6) if

(C1) $D \subset \mathrm{cl}\mathcal{O}^{+}(x)$ for every $x \in D$.

(C2) for every $x \in D$ there is $u(\cdot) \in \mathcal{U}$ such that the corresponding trajectory of (2.6) satisfies $\mathbb{P}\varphi(t, x, u(\cdot)) \in D$ for all $t \in \mathbb{R}$.

(C3) is maximal (with respect to set inclusion) with the properties (C1) and (C2).

A control set D is called *invariant* if in addition to (C1), (C2) and (C3) following holds

(C4) $\mathrm{cl}D = \mathrm{cl}\mathcal{O}^{+}(x)$ for every $x \in D$

Every other control set is called *variant*.

If we have a state space decomposition of the semilinear control system (2.1) into singular subspaces W_i, we can introduce for the projected control systems (2.3) the system groups $\mathcal{G}_i$ and the systems semigroups $\mathcal{S}_i$ through the fundamental solutions of (2.3) on the singular subspaces. These 'new' system groups $\mathcal{G}_i$ act in the same way on the projective singular subspaces $\mathbb{P}(W_i)$ as the original system group $\mathcal{G}$ does. In fact, for every element $g_i \in \mathcal{G}_i$ there exists an element $g \in \mathcal{G}$, such that g_i is the restriction of g on W_i resp. $\mathbb{P}(W_i)$. So we do not make a notational difference between $\mathcal{G}_i$ and $\mathcal{G}$ and between $\mathcal{S}_i$ and $\mathcal{S}$. Therefore, the orbits of a point $x \in \mathbb{P}(W_i)$ induced by the action of $\mathcal{S}$ and $\mathcal{S}_i$ are identical, and we only use the first notation. Because of this, it is also clear, that the control sets of the restricted systems (2.7) are actually control sets for the main system (2.6) and that a control set of the main system, which lies in a projective singular subspace, is a control set for the induced system on this projective subspace.

3. The Floquet spectrum

An 'inner' approximation of the Lyapunov spectrum can be obtained by considering the Floquet spectrum of the system (2.1). We introduce the following abbreviation, which will be used in this section frequently. We define

$$\mathcal{U}_{pc} = \{u : \mathbb{R} \to \mathrm{U} : u(\cdot) \,\text{piecewise constant}\}$$

the set of piecewise constant control functions, which is a subset of $\mathcal{U}$. Every element $g \in \mathcal{S}_\tau, \tau \geq 0$ is associated with a piecewise constant control function $u_{g,\tau}$. Given $g = \exp(t_n A(u_n)) \ldots \exp(t_0 A(u_o))$ for $n \in \mathbb{N}$ with $u_n \in \mathrm{U}$ and $t_n > 0$, such that $\sum_{i=0}^n t_i = \tau$, we define the corresponding (non-unique) control via

$$u_{g,\tau}(t) := u_i$$

for $t \in [t_i, t_{i+1}]$, extended τ-periodically to $\mathbb{R}$.

The Floquet spectrum describes the Lyapunov spectrum of periodic solutions of the semi-linear control system, if we apply piecewise continuous control functions. As we have seen, the periodic trajectories are closely related to the systems semigroup $\mathcal{S}$. The main idea behind the Floquet spectrum is, that we can describe periodic trajectories of the projected control system in a different way. A trajectory $\mathbb{P}\varphi(\cdot, \mathbb{P}x, u(\cdot))$ is a *periodic* trajectory iff there exists $\tau \geq 0$ and $\mu \in \mathbb{R}$ with

$$\varphi(\tau, x, u(\cdot)) = \mu x$$

for a $x \in \mathbb{P}^{-1}(\mathbb{P}x)$. Thus x is an eigenvector for a real eigenvalue μ of the linear time-τ solution. If the main system fulfills a non-degeneracy property (the accessibility rank condition), the systems semigroup $\mathcal{S}$ can be used to describe control sets with nonvoid interior. Here we give a short survey of the results of Colonius and Kliemann, for further information see [**?**] (Colonius et al., 1996).

We say, that the control system (2.6) on projective space satisfies the accessibility rank condition (ARC), if

$$\dim \mathcal{LA}\{h(\cdot, u) : u \in \mathrm{U}\}(p) = d - 1 \text{ for all}\, p \in \mathbb{P}^{d-1}. \tag{ARC}$$

Here $h(\cdot, u)$ denotes the vector field of the projected system on $\mathbb{P}^{d-1}$ as defined in (2.6) for constant $u \in \mathrm{U}$, $\mathcal{LA}$ denotes the Lie algebra generated by these vector field, and $\dim\mathcal{LA}(p)$ is the dimension of the distribution generated by $\mathcal{LA}$ in the tangent space $T_p\mathbb{P}^{d-1}$ of $\mathbb{P}^{d-1}$ at the point $p \in \mathbb{P}^{d-1}$. The assumption (ARC) is equivalent to the requirement that the system (2.6) is locally accessible, see e.g. [**?**] (Isidori, 1989). Furthermore, the accessibility rank condition implies, that the interior $\mathrm{int}\mathcal{S}_{\leq t}$ of $\mathcal{S}_{\leq t}$ in $\mathcal{G}$ is non-empty. For $g \in \mathrm{GL}(d, \mathbb{R})$ let $\mathrm{spec}(g)$ be the spectrum of g and $E(\lambda)$ the (generalized) eigenspace of $\lambda \in \mathrm{spec}(g)$. We introduce the notation

$$\begin{aligned} V &= \bigcup\{E(\lambda) : \lambda \in \mathrm{spec}(g), g \in \mathrm{int}\mathcal{S}\} \subset \mathbb{R}^d \text{ , and} \\ \mathbb{P}(V) &= \{\mathbb{P}(E(\lambda)) : \lambda \in \mathrm{spec}(g), g \in \mathrm{int}\mathcal{S}\} \subset \mathbb{P} \end{aligned}$$

The following result was proved in [**?**] (Colonius et al., 1996), Theorem 3.10.

Theorem 3.1. Let the accessibility rank condition (ARC) be satisfied.

(i) There are L control sets $D_1, \dots, D_L$, $1 \leq L \leq d$ with nonvoid interior, which we call *main* control sets. The connected components of $\mathbb{P}(V)$ are the interiors of the main control sets.

(ii) The main control sets are linearly ordered by
$D_i \prec D_j$ if there exists $x_i \in D_i, x_j \in D_j, t > 0$ and $g \in \mathcal{S}_t$ with $gx_i = x_j$.
We enumerate the main control sets such that $D_1 \prec \dots \prec D_L$.

(iii) For every $g \in \mathrm{int}\mathcal{S}_{\leq t}$, and every main control sets D there exists $\mu \in \mathrm{spec}(g)$ with $\mathbb{P}(E(\mu)) \subset \mathrm{int}D$. Furthermore, if $\mu, \mu' \in \mathrm{spec}(g)$ with $\mathrm{Re}\mu < \mathrm{Re}\mu'$, and $\mathbb{P}(E(\mu)) \subset \mathrm{int}D, \mathbb{P}(E(\mu')) \subset \mathrm{int}D'$, then $D \leq D'$.

(iv) If for $g \in \mathcal{S}$ and $\mu \in \mathrm{spec}(g)$ one has $\mathbb{P}(E(\mu)) \subset \mathrm{int}D$ for some main control set D, then $g \in \mathrm{int}\mathcal{S}_{\leq t}$ for some $t > 0$.

This theorem suggests following definition.

Definition 3.2. Let D be a control set with nonvoid interior of the projected control system (2.6). The Floquet spectrum over D is defined as

$$\Sigma_{Fl}(D) = \left\{ \lambda(u(\cdot), x) : \begin{array}{l} x \in \mathrm{int}D \text{ and } u(\cdot) \in \mathcal{U}_{pc} \text{ is } \tau\text{-periodic} \\ \text{for a } \tau \geq 0, \text{such that } \; \mathbb{P}\varphi(\tau, x, u(\cdot)) = x \end{array} \right\}$$

The Floquet spectrum of the semilinear control system (2.6) is

$$\Sigma_{Fl} = \bigcup \Sigma_{Fl}(D)$$

where the union is taken over all control sets $D \subseteq \mathbb{P}^{d-1}$ with nonvoid interior.

It is clear, that $\Sigma_{Fl} \subseteq \Sigma_{Ly}$, which justifies the notion of an 'inner' approximation of the Lyapunov spectrum.

If the accessibility rank condition (ARC) is satisfied, many results concerning the Floquet spectrum are known, see [**?**] (Colonius et al., 1996), section 4. But if our main control system (2.1) is decomposable into singular subspaces, the following lemma states, that the accessibility rank condition can not be satisfied.

Lemma 3.3. Assume, that state space of the semilinear control system (2.1) is decomposable into singular subspaces $W_1, \dots, W_K$ with $K \geq 2$. Then the projectivized control system (2.6) is not locally accessible. Therefore the accessibility rank condition (ARC) is not satisfied.

Proof. Because of the invariance of the projectivized control system (2.6), the orbit of a point $x \in \mathbb{P}(W_i)$ lies entirely in $\mathbb{P}(W_i)$. Therefore, the orbit has void interior, and the control system (2.6) cannot be locally accessible. □

Our aim is to specify the Floquet spectrum of the semilinear control system with the help of the singular subspaces. Because of the absence of local accessibility, many tools of geometric control theory are not applicable in this case. However the Floquet spectrum is closely related to control sets with nonvoid interior. The idea is to describe these control sets with the help of the restricted control systems (2.7). For this analysis, we have to introduce projection maps on projective subspaces $\mathbb{P}(W_i)$.

Recall, that if we have a decomposition $\mathbb{R}^d = V \oplus W$ of $\mathbb{R}^d$ into linear subspaces V and W with $V, W \neq \{0\}$, we can write every element $x \in \mathbb{R}^d$ as $x = v + w$ with unique $v \in V$ and $w \in W$. The projection of $\mathbb{R}^d$ (along W) onto the linear subspace V is defined via

$$\begin{array}{rccc} pr_V: & \mathbb{R}^d & \to & V \\ & v+w & \mapsto & v \end{array}$$

The projection pr_V has following properties

$$pr_V \cdot pr_V = pr_V \qquad \text{and} \qquad pr_V(V) = V$$

We transfer the concept of projection to the projective space $\mathbb{P}^{d-1}$.

Definition 3.4. Let $\mathbb{R}^d = V \oplus W$ be a decomposition of $\mathbb{R}^d$ into linear subspaces V and W with $V, W \neq \{0\}$. The *projection* of $\mathbb{P}^{d-1}$ onto $\mathbb{P}(V)$ is the map

$$\begin{array}{rccc} \mathbb{P}_V: & \mathbb{P}(\mathbb{R}^d \setminus W) & \to & \mathbb{P}(V) \\ & x & \mapsto & \mathbb{P}(pr_V(\mathbb{P}^{-1}(x))). \end{array}$$

Remark 3.5. i) Note, that the projection $\mathbb{P}_V$ is well-defined, since

$$\mathbb{P}(\mathbb{R}^d \setminus W) = \mathbb{P}(\{x \in \mathbb{R}^d : pr_V x \neq 0\}).$$

We denote the domain of definition by $\mathrm{Dom}(\mathbb{P}_V)$, which is an open subset of $\mathbb{P}^{d-1}$. In particular, subsets of $\mathbb{P}(\mathbb{R}^d \setminus W)$ which are open with respect to the induced topology on $\mathbb{P}(\mathbb{R}^d \setminus W)$, are open subsets in $\mathbb{P}^{d-1}$.

ii) The name *projection* for $\mathbb{P}_V$ comes from following properties

$$\mathbb{P}_V \circ \mathbb{P}_V = \mathbb{P}_V \qquad \text{and} \qquad \mathbb{P}_V(\mathbb{P}(V)) = \mathbb{P}(V).$$

iii) The projection $\mathbb{P}_V$ is an open and closed, differentiable function.

If the state space ofsemilinear control system (2.1) is decomposable into nontrivial singular subspaces $W_1, \ldots, W_K, K \geq 1$, we denote the projection on $\mathbb{P}(W_i)$ by $\mathbb{P}_i$ instead of $\mathbb{P}_{W_i}$. Note, that

$$\mathrm{Dom}(\mathbb{P}_i) = \mathbb{P}(\mathbb{R}^d \setminus \bigoplus_{j=1,\, j\neq i}^{K} W_j).$$

Remark 3.6. If the semilinear control system (2.1) is decomposable into singular subspaces $W_1, \ldots, W_K$, then for every $x \in \mathrm{Dom}(\mathbb{P}_i)$ and every $i = 1, \ldots, K$ it holds that:

$$\mathbb{P}_i(\mathbb{P}\varphi(t, x, u(\cdot))) = \mathbb{P}\varphi(t, \mathbb{P}_i(x), u(\cdot)) = \mathbb{P}\varphi_i(t, \mathbb{P}_i(x), u(\cdot)) \text{ for all } t \in \mathbb{R}. \tag{3.1}$$

The next proposition shows, that the projections of control sets onto projective subspaces ly in control sets.

Proposition 3.7. Assume, that the state space of the semilinear control system (2.1) is decomposable into singular subspaces $W_1, \ldots, W_K, K \geq 1$. Let $D \subset \mathbb{P}^{d-1}$ be a control set of the projectivized control system (2.6).

Then there exists a nonempty index set $\mathcal{I}(D) \subseteq \{1, \ldots, K\}$, such that for all $i \in \mathcal{I}(D)$:

$$D \subseteq \mathrm{Dom}(\mathbb{P}_i) \tag{3.2}$$

and the set

$$\mathbb{P}_i(D) \subseteq \mathbb{P}(W_i) \tag{3.3}$$

satisfies the conditions (C1) and (C2) for control sets. In particular, $\mathbb{P}_i(D)$ is contained in a control set.

If D is an invariant control set, then in addition to this, $\mathbb{P}_i(D)$ satisfies condition (C4).

Proof. Suppose that $D \cap \mathrm{Dom}(\mathbb{P}_i) = \emptyset$ for all $i \in \{1, \ldots, K\}$ and let $\mathbb{P}x = \mathbb{P}(\sum_{j=1}^K x_j) \in D$ for $x_j \in W_j$. This implies $\mathbb{P}x \in \mathbb{P}(\bigoplus_{j=1, j\neq i}^K W_j \setminus \{0\})$ for all $i = 1, \ldots, K$. Therefore $x_i = 0$ for all $i = 1, \ldots, K$, which contradicts $\sum_{j=1}^K x_j \neq 0$. Hence there is at least one $i \in \{1, \ldots, K\}$ with $D \cap \mathrm{Dom}(\mathbb{P}_i) \neq 0$.

Consider all indices $i_i, \ldots, i_l \subseteq \{1, \ldots, K\}$ with

$$D \cap \mathrm{Dom}(\mathbb{P}_{i_m}) \neq \emptyset$$

for $m = 1, \ldots, l$. We define

$$\mathcal{I}(D) := \{i_1, \ldots, i_l\}$$

and prove, that this is the desired index set. First we have to show the inclusion (3.2). Assume, that there exists a $x \in D$ and an $i \in \mathcal{I}(D)$ with $x \notin \mathrm{Dom}(\mathbb{P}_i)$. Since $\mathbb{P}^{d-1} \setminus \mathrm{Dom}(\mathbb{P}_i) = \mathbb{P}(\bigoplus_{j=1, j\neq i}^K W_j)$ it would follow, that D is a subset of $\mathbb{P}(\bigoplus_{j=1, j\neq i_m}^K W_j)$. But this contradicts the construction of $\mathcal{I}(D)$.

Because (3.2) is valid, the control set D lies in the domain of definition of the maps $\mathbb{P}_i, i \in \mathcal{I}(D)$. Thus we can project D onto the subspaces $\mathbb{P}(W_i)$, and we get the sets $\mathbb{P}_i(D)$. Next, we have to show, that for every $x_i \in \mathbb{P}_i(D)$ there exists an $u(\cdot) \in \mathcal{U}$, such that $\mathbb{P}\varphi(t, x_i, u(\cdot)) \in D$ for all $t > 0$. Now D is a control set, and therefore for every $x \in D$ there is an $u(\cdot) \in \mathcal{U}$ with $\mathbb{P}\varphi(t, x, u(\cdot)) \in D$ for all $t > 0$. Since $\mathbb{P}\varphi(t, x, u(\cdot))$ lies in the domain of definition of $\mathbb{P}_i$, we have

$$\mathbb{P}\varphi(t, \mathbb{P}_i(x), u(\cdot)) = \mathbb{P}_i(\mathbb{P}\varphi(t, x, u(\cdot))) \in \mathbb{P}_i(D)$$

for all $t > 0$, and (C1) is shown.

Next, we show, that for all $x_i \in \mathbb{P}_i(D)$ following inclusion holds:

$$\mathbb{P}_i(D) \subseteq \mathrm{cl}\mathcal{O}^+(x_i)$$

We take $x_i, y_i \in D$. By definition of $\mathbb{P}_i(D)$ there exists $x, y \in D$ with $x_i = \mathbb{P}_i(x)$ and $y_i = \mathbb{P}_i(y)$. Because D is a control set, there exists a sequences $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ with

$$\lim_{n\to\infty} g_n x = y$$

Using the equality $\mathbb{P}_i(g_n x) = g_n \mathbb{P}_i(x)$ and the continuity of $\mathbb{P}_i$, we get

$$y_i = \mathbb{P}_i(y) = \mathbb{P}_i(\lim_{n\to\infty} g_n x) = \lim_{n\to\infty} g_n \mathbb{P}_i(x) = \lim_{n\to\infty} g_n x_i.$$

That means, that $y_i \in \mathrm{cl}\mathcal{O}^+(x_i)$, and that $\mathbb{P}_i(D) \subseteq \mathrm{cl}\mathcal{O}^+(x_i)$. We have shown property (C2) for control sets and as $\mathbb{P}_i(D)$ satisfies property (C1), as shown before, it is part of a control set $D_i \subset \mathbb{P}(W_i)$, see [**?**] (Häckl, 1996), Lemma 1.2.5.

If D is an invariant control set, we have to show that for all $x_i \in D$ the following equality holds:

$$\mathrm{cl}\mathbb{P}_i(D) = \mathrm{cl}\mathcal{O}^+(x_i).$$

As we have seen before, $\mathbb{P}_i(D)$ is a subset of $\mathrm{cl}\mathcal{O}^+(x_i)$, which implies the inclusion $\mathrm{cl}\mathbb{P}_i(D) \subseteq \mathrm{cl}\mathcal{O}^+(x_i)$. We now show the other inclusion. Given $x_i \in D$ and $y_i \in \mathcal{O}^+(x_i)$, there exists a $g \in \mathcal{S}$ with

$$y_i = g x_i.$$

Because x_i lies in the projection of the control set D, there is a $x \in D$ with $\mathbb{P}_i(x) = x_i$. We define $y := gx$, and because D is an invariant control set, y lies in $\mathrm{cl}D$. Thus there is a sequence $\{z_n\}_{n\in\mathbb{N}} \subset D$ with $\lim_{n\to\infty} z_n = y$. Now the projection of this sequence on the projective singular subspace $\mathbb{P}(W_i)$, defines a sequence $\{\mathbb{P}_i(z_n)\}_{n\in\mathbb{N}} \subset \mathbb{P}(W_i)$ which converges to y_i for $n \to \infty$. This means, that $y_i \in \mathrm{cl}\mathbb{P}_i(D)$ which implies that $\mathcal{O}^+(x_i) \subseteq \mathrm{cl}\mathbb{P}_i(D)$ for all $x_i \in D$. Because this implies $\mathrm{cl}\mathcal{O}^+(x_i) \subseteq \mathrm{cl}\mathbb{P}_i(D)$, we have shown property (C4). □

It is clear, that if D is a subset of a singular subspace $\mathbb{P}(W_i)$, the only applicable projection of D is the projection $\mathbb{P}_i$. In this case $\mathcal{I}(D) = \{i\}$. More generally, if there are $i_1, \dots, i_l \subset \{1, \dots, K\}$ such that $D \subseteq \mathbb{P}(W_{i_1} \oplus \dots \oplus W_{i_l})$, then $\mathcal{I}(D) \subseteq \{i_1, \dots, i_l\}$.

Control sets with nonvoid interior, which are of special interest for us, can be projected on every projective singular subspace $\mathbb{P}(W_i)$.

Corollary 3.8. Assume, that the state space of the semilinear control system (2.1) is decomposable into singular subspaces $W_1, \dots, W_K, K \geq 1$. Let $D \subset \mathbb{P}^{d-1}$ be a control set with nonvoid interior. Then we can project D onto every projective singular subspace $\mathbb{P}(W_i), i = 1, \dots, K$.

In addition to this $\mathbb{P}_i(D) \subset \mathbb{P}(W_i)$ are sets with nonvoid interior, with respect to the induced topology on $\mathbb{P}(W_i)$.

Proof. Assume that $\mathcal{I}(D) \neq \{1, \dots, K\}$ where $\mathcal{I}(D)$ denotes the index set introduced in Proposition 3.7. Then there exists $i \in \{1, \dots K\}$ with $D \cap \mathrm{Dom}(\mathbb{P}_i) = \emptyset$. But this implies $D \subseteq \mathbb{P}(\sum_{j=1, j\neq i}^{K} W_j)$, and D would have void interior. This shows the first assertion. Because the projections $\mathbb{P}_i$ are open, the interior of D is mapped to open sets with respect to the induced topology on the projective singular subspaces. □

The projection of control sets allows us to characterize the Floquet spectrum of a control set $D \subset \mathbb{P}^{d-1}$ with nonvoid interior. Here we use the Floquet spectrum of control sets, which ly in the projected singular subspaces and have nonvoid interior in the relative topology. We define the Floquet spectrum of such a control set $D_i \subset \mathbb{P}(W_i)$ is defined as

$$\Sigma_{Fl}(D_i) = \left\{ \lambda(u(\cdot), x) : \begin{array}{l} x_i \in \mathrm{int}_{\mathbb{P}(W_i)} D_i \text{ and } u(\cdot) \in \mathcal{U}_{pc} \text{ is } \tau\text{-periodic} \\ \text{for a } \tau \geq 0, \text{such that } \mathbb{P}\varphi_i(\tau, x_i, u(\cdot)) = x_i \end{array} \right\}$$

Here $\mathrm{int}_{\mathbb{P}(W_i)} D_i$ denotes the interior of D_i relative to $\mathbb{P}(W_i)$.

Theorem 3.9. Assume that the state space of the semilinear control system (2.1) is decomposable into singular subspaces $W_1, \ldots, W_K$. Let $D \subset \mathbb{P}^{d-1}$ be a control set with nonvoid interior. Then

$$\Sigma_{Fl}(D) \subseteq \bigcap_{i=1}^{K} \Sigma_{Fl}(D_i)$$

where the $D_i \subseteq \mathbb{P}(W_i)$ are control sets with nonvoid interior in the relative topology on $\mathbb{P}(W_i)$ with $\mathbb{P}_i(D) \subseteq D_i$.

Proof. Let $\mathbb{P}x \in \mathrm{int}D$ and take a τ-periodic piecewise constant control function $u(\cdot) \in \mathcal{U}_{pc}$, such that the associated trajectory $\mathbb{P}\varphi(t, \mathbb{P}x, u(\cdot)), t \in [0, \tau]$ is τ-periodic. We showed in Corollary 3.8., that we can project D on every projective singular subspace $\mathbb{P}(W_i)$, and that $\mathbb{P}_i(D)$ is part of a control set $D_i \subseteq \mathbb{P}(W_i)$ with nonvoid interior in the induced topology on $\mathbb{P}(W_i)$. Since $\mathbb{P}x$ lies in the interior of the control set D, and because $\mathbb{P}_i$ is an open map, the projection $\mathbb{P}_i(\mathbb{P}x)$ lies in the interior of $\mathbb{P}_i(D)$.

Now the projection of the trajectory $\mathbb{P}\varphi(\cdot, \mathbb{P}x, u(\cdot))$ on $\mathbb{P}(W_i)$ is τ-periodic, because

$$\mathbb{P}_i(\mathbb{P}x) = \mathbb{P}_i(\mathbb{P}\varphi(\tau, \mathbb{P}x, u(\cdot)) = \mathbb{P}\varphi_i(\tau, \mathbb{P}_i(\mathbb{P}x), u(\cdot))$$

The Lyapunov exponent of $(u(\cdot), \mathbb{P}x)$ is given by the time-τ solution operator $g_{u(\cdot),\tau}$. There exists an $\mu \in \mathbb{R}$ such that every $x \in \mathbb{P}^{-1}(\mathbb{P}x)$ is an eigenvector of $g_{u(\cdot),\tau}$, that means $g_{u(\cdot),\tau} x = \mu x$. By definition, $g_{u(\cdot),\tau}$ is an element of the systems semigroup $\mathcal{S}$, and it follows

$$g_{u(\cdot),\tau}\Big|_{W_i} \in \mathcal{S}_i$$

for every $i = 1, \ldots, K$. Thus $pr_i(x) \in W_i$ is an eigenvector of $g_{u(\cdot),\tau}\Big|_{W_i}$ with eigenvalue μ. Therefore $\lambda(u(\cdot), \mathbb{P}_i(\mathbb{P}x)) = \lambda(u(\cdot), \mathbb{P}(pr_i(x))) = \frac{1}{\tau} \log |\mu|$ and it follows that

$$\lambda(u(\cdot), \mathbb{P}x) = \lambda(u(\cdot), \mathbb{P}_i(\mathbb{P}x))$$

Since this equality holds for all $i = 1, \ldots, K$, the assertion follows. □

If the accessibility rank condition (ARC) holds on every projective singular subspace, we can describe the Floquet spectrum with the help of the main control sets on the projected singular subspaces .

Corollary 3.10. Assume that the state space of the semilinear control system (2.1) is decomposable into singular subspaces $W_1, \ldots, W_K$. Let the accessibility rank condition (ARC) be satisfied for the restricted control systems (2.7). Then the Floquet spectrum of the semilinear control system (2.1) is

$$\Sigma_{Fl} \subseteq \bigcup_{i=1}^{K} \bigcup_{j=1}^{L_i} \Sigma_{Fl}(D_i^j)$$

where the $D_i^j \subseteq \mathbb{P}(W_i)$ are the main control sets of the projected control systems (2.7) on $\mathbb{P}(W_i)$.

Proof. Since the accessibility rank condition holds on the projective singular subspaces $\mathbb{P}(W_i)$, we can apply Theorem 3.1. to the restricted control systems (2.7). For every $i = 1, \ldots, K$ there exist main control sets $D_i^1, \ldots, D_i^{L_i} \subseteq \mathbb{P}(W_i)$ with nonvoid interior in the relative topology on $\mathbb{P}(W_i)$. Let D be a control set in $\mathbb{P}^{d-1}$ with nonvoid interior. Since the projection $\mathbb{P}_i$ is an open map, the image $\mathbb{P}_i(D)$ has nonvoid interior in $\mathbb{P}(W_i)$ in the relative topology. Because $\mathbb{P}_i(D)$ is a subset of a control set, it is part of a main control set in $\mathbb{P}(W_i)$ for $i = 1, \ldots, K$. Thus, a Floquet exponent which belongs to the control set D in the main system is a Floquet exponent of a main control set on a projected singular subspace. □

4. The Morse spectrum

In the previous section, we have looked at semilinear control systems. In this section we are interested in the following special case, namely bilinear control systems.

$$\begin{array}{ll} \dot{x}(t) = A_0 x(t) + \sum_{i=1}^m u_i(t) A_i x(t), \qquad t \in \mathbb{R}, \qquad x(0) = x_0 \in \mathbb{R}^d, \\ u(\cdot) \in \mathcal{U}_{loc} := \{u(\cdot) : \mathbb{R} \to \mathrm{U} : u(\cdot) \text{ locally integrable}\}, \end{array} \tag{4.1}$$

where $\mathrm{U} \subset \mathbb{R}^m$ is convex and compact, and $A_i \in \mathrm{Mat}(d, \mathbb{R})$ for all $i = 0, \ldots, m$. There is an associated control system on projective space

$$\begin{array}{ll} \dot{p}(t) = h(p, u(\cdot)), \qquad t \in \mathbb{R}, \qquad p(0) = p_0 \in \mathbb{P}^{d-1}, \\ u(\cdot) \in \mathcal{U}_{loc} := \{u(\cdot) : \mathbb{R} \to \mathrm{U} : u(\cdot) \text{ local integrable}\}, \end{array} \tag{4.2}$$

Here the angular component is $h(p, u) = h_0(p) + \sum_{i=1}^m u_i h_i(p)$ with $h_j(p) = [A_j - p^T A_j p Id]p$ for $j = 0, \ldots, m$.

For the bilinear control systems, the assertions of the previous section hold too. The reason for restricting ourselves to this subclass of semilinear control systems is, that in this case, we are now able to define a continuous control flow for the bilinear system. The flow point of view allows us to use concepts and techniques from topological dynamics for the analysis of the Lyapunov spectrum.

We will denote in this section elements of the projective space $\mathbb{P}^{d-1}$ by $\mathbb{P}x, \mathbb{P}y, \ldots$. This notation is useful, if we have to represent elements of the projective space by elements of $\mathbb{R}^d$.

We look at the associated *control flow*, which is given by

$$\begin{array}{rlcl} \Phi: & \mathbb{R}\times\mathcal{U}\times\mathbb{R}^d & \longrightarrow & \mathcal{U}\times\mathbb{R}^d \\ & (t,u(\cdot),x) & \longmapsto & \Phi(t,u(\cdot),x)=(\theta_t(u(\cdot)),\varphi(t,x,u(\cdot)) \end{array} \tag{4.3}$$

where $\theta_t(u(\cdot)) := u(t+\cdot)$ is the usual shift by t. If one endows $\mathcal{U}$ with the weak* topology of $L^\infty(\mathbb{R},\mathbb{R}^m)^*$, then Φ becomes a continuous flow, see [?] (Colonius et al., 1996) for details. With the control flow on $\mathcal{U}\times\mathbb{R}^d$ we associate the *projectivized* flow

$$\begin{array}{rlcl} \mathbb{P}\Phi: & \mathbb{R}\times\mathcal{U}\times\mathbb{P}^{d-1} & \longrightarrow & \mathcal{U}\times\mathbb{P}^{d-1} \\ & (t,u(\cdot),\mathbb{P}x) & \longmapsto & (\theta_t(u(\cdot)),\mathbb{P}\varphi(t,\mathbb{P}x,u(\cdot)) \end{array} \tag{4.4}$$

Note, that the control flow Φ is a *linear flow* on the vector bundle $\pi:\mathcal{U}\times\mathbb{R}^d \to \mathcal{U}$. This means, that

$$\begin{gathered} \pi(\Phi(t,u(\cdot),x_1)) = \pi(\Phi(t,u(\cdot),x_2)), \text{ and} \\ \Phi(t,\alpha((u(\cdot),x_1)+(u(\cdot),x_2))) = \alpha\Phi(t,u(\cdot),x_1)+\alpha\Phi(t,u(\cdot),x_2)) \end{gathered}$$

for all $(u(\cdot),x_1),(u(\cdot),x_2)\in\mathcal{U}\times\mathbb{R}^d$ and $\alpha\in\mathbb{R}$. For the definition of vector bundles compare e.g. [?] (Karoubi, 1978). The projectivized flow $\mathbb{P}\Phi$ can also interpreted as the projectivization of the linear flow on the projective bundle $\mathbb{P}\pi:\mathcal{U}\times\mathbb{P}^{d-1}\to\mathcal{U}$.

For $\varepsilon,T>0$ an (ε,T)*-chain* ζ of $\mathbb{P}\Phi$ is given by $n\in\mathbb{N}, T_0,\ldots,T_{n-1}\geq T$, and

$$(u_0(\cdot),\mathbb{P}x_0),\ldots,(u_n(\cdot),\mathbb{P}x_n)\in\mathcal{U}\times\mathbb{P}^{d-1}$$

with

$$d\left(\mathbb{P}\Phi(T_i,u_i(\cdot),\mathbb{P}x_i),(u_{i+1}(\cdot),\mathbb{P}x_{i+1})\right)<\varepsilon$$

for $i=0,\ldots,n-1$. Here $d(\cdot,\cdot)$ denotes a metric on $\mathcal{U}\times\mathbb{P}^{d-1}$, see [?] (Colonius et al. 1996), Lemma 2.1.

We denote the chain recurrent set of $\mathbb{P}\Phi$ by $\mathcal{R}$, i.e. $\mathcal{R}$ is the set of points $(u(\cdot),x)$ in $\mathcal{U}\times\mathbb{P}^{d-1}$ such that for every $\varepsilon>0, T>0$ there exists an (ε,T)-chain ζ with $(u_0(\cdot),\mathbb{P}x_0)=(u_n(\cdot),\mathbb{P}x_n)=(u(\cdot),\mathbb{P}x)$. The set $\mathcal{R}$ has R connected components $\mathcal{M}_1,\ldots,\mathcal{M}_R$, $(1\leq R\leq d)$, the *Morse sets* of $\mathbb{P}\Phi$. Their projections $E_j:=\pi_{\mathbb{P}}\mathcal{M}_j\subset\mathbb{P}^{d-1}$ are the chain control sets of (4.2), see [?] (Colonius et al., 1993), Theorem 4.9.

Define the *finite time exponential growth rate* of a chain ζ (or *chain exponent*) by

$$\lambda(\zeta)=\left(\sum_{i=0}^{n-1}T_i\right)^{-1}\cdot\left(\sum_{i=0}^{n-1}\log|\varphi(T_i,x_i,u_i(\cdot))|-\log|x_i|\right)$$

with $x_i\in\mathbb{P}^{-1}(\mathbb{P}x_i)$, $i\in\{0,\ldots,n\}$. The Morse spectrum of Φ in $\mathcal{M}_j$ is given by

$$\Sigma_{Mo}(\mathcal{M}_j):=\left\{\lambda\in\mathbb{R}: \begin{array}{l} \exists\varepsilon^k\to 0, T^k\to\infty \text{ and } (\varepsilon^k,T^k)\text{-chains } \zeta^k \\ \text{in } \mathcal{M}_j \text{ with } \lambda(\zeta^k)\to\lambda \text{ for } k\to\infty \end{array}\right\} \tag{4.5}$$

The Morse spectrum of the bilinear control system (4.1) is defined as

$$\Sigma_{Mo}:=\bigcup_{j=1}^{R}\Sigma_{Mo}(\mathcal{M}_j) \tag{4.6}$$

where the union is taken over all chain recurrent components of $\mathcal{R}$. In the following theorem we collect some basic facts about the Morse spectrum on Φ .

Theorem 4.1. (i) For $j = 1, \dots, R$, $\Sigma_{Mo}(\mathcal{M}_j)$ is a compact interval. i.e. it has the form $\Sigma_{Mo}(\mathcal{M}_j) = [\kappa^*(\mathcal{M}_j), \kappa(\mathcal{M}_j)]$.

(ii) $\Sigma_{Ly} \subseteq \Sigma_{Mo}$ and $\kappa^*(\mathcal{M}_j), \kappa(\mathcal{M}_j)$ are actually Lyapunov exponents for some $(u_j^*(\cdot), \mathbb{P}x_j^*)$, $(u_j(\cdot), \mathbb{P}x_j) \in \mathcal{M}_j, j = 1, \dots, R$.

(iii) $\mathcal{U} \times \mathbb{R}^d = \bigoplus_{j=1}^{R} \mathcal{V}_j$ (Whitney sum), where

$$\mathcal{V}_j := \{(u(\cdot), x) \in \mathcal{U} \times \mathbb{R}^d \ : x \neq 0 \Rightarrow (u(\cdot), \mathbb{P}(x)) \in \mathcal{M}_j\}$$

Proof. (i) follows from [**?**] (Colonius et al. ,1996), Theorem 3.6. Assertion (ii) follows from [**?**], Theorem 3.7 and 4.6. For assertion (iii) see [**?**], Theorem 3.1 and note that the base space $\mathcal{U}$ for Φ is chain recurrent by [**?**] (Colonius et al., 1993), Lemma 4.5 □

In the previous section we have seen, that we can describe the Floquet exponents of the main system by the Floquet exponents of the restricted lower dimensional systems, which we get, if our system is decomposable into singular subspaces $W_1, \dots, W_K$. Now we want to show a similar statement for the Morse spectrum of the bilinear control system (4.1). Here the objects of interest are the Morse sets $\mathcal{M}_j$, instead of control sets for the Floquet spectrum.

For analysing the Floquet spectrum, we have projected (periodic) trajectories down to the projected singular subspaces $\mathbb{P}(W_i)$. An analogous attempt for the Morse spectrum is not advisable. Control sets with nonvoid interior can not overlap with projected singular subspaces $\mathbb{P}(W_i)$, because of the invariance of the control system on these projected subspaces. Therefore we can project these control sets on every projective invariant subspace. In contrast to this, chain control sets, which are the projection of the Morse sets on the projective space, can overlap with the projected singular subspaces $\mathbb{P}(W_i)$, because here small jumps out of the invariant projected subspaces are allowed. This makes it difficult to define the projection of an (ε, T)-chain onto the projected subspaces. It may occur, that some points of the chain do *not* ly in the domain of definition of the projection $\mathbb{P}_i$ onto some $\mathbb{P}(W_i)$. Thus we cannot project these points of the chain onto this projective singular subspace. Therefore we do not in general get an (ε, T)-chain on $\mathbb{P}(W_i)$ by only projecting down (ε, T)-chains in $\mathbb{P}^{d-1}$.

Instead of projecting chain control sets on projective subspaces, we look at the Morse sets on the projectivized subbundles that we get, if we restrict the main control flow to the singular subspaces. We will show, that we can represent Morse sets of the main system with the help of Morse sets which belong to the restricted systems.

For doing this, we have to introduce some notation. Assume, that the bilinear control system is decomposable into singular subspaces $W_1, \dots, W_K$. If we define the associated control systems on these subspaces like in (2.3), we get actually *bilinear* control systems on these subspaces. On the restricted control systems we define linear flows Φ_i on the linear vector bundle $\pi : \mathcal{U} \times W_i \to \mathcal{U}$, by restriction of the linear flow Φ.

$$\begin{array}{rcl} \Phi_i : \mathbb{R} \times \mathcal{U} \times W_i & \longrightarrow & \mathcal{U} \times W_i \\ (t, u(\cdot), x_i) & \longmapsto & (\theta_t(u(\cdot)), \varphi_i(t, x_i, u(\cdot))). \end{array}$$

Note, that these restricted control flows are indeed the control flows induced by the dynamic of the bilinear control systems on the W_i. Similarly we define the projectivized flows on the projective bundle $\mathbb{P}\pi_i : \mathcal{U} \times \mathbb{P}(W_i) \to \mathcal{U}$, by

$$\begin{array}{rcl} \mathbb{P}\Phi_i : \mathbb{R} \times \mathcal{U} \times \mathbb{P}(W_i) & \longrightarrow & \mathcal{U} \times \mathbb{P}(W_i) \\ (t, u(\cdot), \mathbb{P}x_i) & \longmapsto & (\theta_t(u(\cdot), \mathbb{P}\varphi_i(t, \mathbb{P}x_i, u(\cdot)) \end{array}$$

For every $i \in \{1, \ldots, K\}$ the chain recurrent sets $\mathcal{R}_i$ of the flow $\mathbb{P}\Phi_i$ have R_i connected components, which we denote by $\mathcal{M}_i^1, \ldots, \mathcal{M}_i^{R_i}$ with $R_i \in \{1, \ldots, \dim W_i\}$.

By applying Theorem 4.1. to the restricted system, we get the following results:

(i) For $j = 1, \ldots, R_i$, $\Sigma_{Mo}(\mathcal{M}_i^j)$ is a compact interval, i.e. it has the form $\Sigma_{Mo}(\mathcal{M}_i^j) = [\kappa^*(\mathcal{M}_i^j), \kappa(\mathcal{M}_i^j)]$.

(ii) $\Sigma_{Ly} \subseteq \Sigma_{Mo}$ and $\kappa^*(\mathcal{M}_i^j), \kappa(\mathcal{M}_i^j)$ are actually Lyapunov exponents for some $(u_i^{*j}(\cdot), \mathbb{P}x_i^{*j})$, $(u_i^j(\cdot), \mathbb{P}x_i^j) \in \mathcal{M}_i^j, j = 1, \ldots, R_i$.

(iii) $\mathcal{U} \times W_i = \bigoplus_{j=1}^{R_i} \mathcal{V}_i^j$ (Whitney sum), where

$$\mathcal{V}_i^j := \{(u(\cdot), x) \in \mathcal{U} \times W_i : x \neq 0 \Rightarrow (u(\cdot), \mathbb{P}(x)) \in \mathcal{M}_i^j\}$$

The next lemma characterizes the Morse sets $\mathcal{M}_1, \ldots, \mathcal{M}_R$ of the main system (4.2) through the Morse sets of the restricted flows $\mathbb{P}\Phi_i$. In the following $\mathcal{P}(M)$ denotes the family of all subsets of a set M.

Lemma 4.2. Assume, that the state space of the bilinear control system (4.1) is decomposable into singular subspaces $W_1, \ldots, W_K$.

(i) For every $i = 1, \ldots, K$ and every $j = 1, \ldots, R_i$ there exists an index $r := r(i, j) \in \{1, \ldots, l\}$ with $\mathcal{M}_i^j \subseteq \mathcal{M}_r$.

(ii) For every $\mathcal{M}_r$ there exists an unique index set

$$\mathcal{I}(\mathcal{M}_r) \subset \mathcal{P}(\{1, \ldots, K\} \times \{1, \ldots, \max_{i=1}^{K} R_i\})$$

with $(i, j) \in \mathcal{I}(\mathcal{M}_r) \Rightarrow j \in \{1, \ldots, R_i\}$, such that the chain recurrent component $\mathcal{M}_r$ can be written as

$$\mathcal{M}_r = \left\{ (u(\cdot), \mathbb{P}x) \in \mathcal{U} \times \mathbb{P}^{d-1} : \begin{array}{l} \text{for all } (i,j) \in \mathcal{I}(\mathcal{M}_r) \text{ it holds:} \\ \mathbb{P}x \in \mathrm{Dom}(\mathbb{P}_i) \Rightarrow (u(\cdot), \mathbb{P}_i(\mathbb{P}x)) \in \mathcal{M}_i^j \end{array} \right\},$$

and its associated vector subbundle as

$$\mathcal{V}_r = \bigoplus_{(i,j) \in \mathcal{I}(\mathcal{M}_r)} \mathcal{V}_i^j.$$

Proof. The sets $\mathcal{M}_i^j \subseteq \mathcal{U} \times \mathbb{P}(W_i)$ are chain transitive components of $\mathbb{P}\Phi_i$. Therefore, they are chain transitive subsets of the projective flow $\mathbb{P}\Phi$, and because the Morse sets $\mathcal{M}_r$ are isolated the assertion (i) holds.

To show the second assertion, we use the fact, that we can decompose the linear vector bundle $\pi : \mathcal{U} \times \mathbb{R}^d \to \mathcal{U}$ in two different ways, namely the decomposition which we get from the Morse sets $\mathcal{M}_1, \ldots, \mathcal{M}_R$ and the decomposition of the Morse sets on the restricted control flows:

$$\mathcal{U} \times \mathbb{R}^d = \bigoplus_{j=1}^{R} \mathcal{V}_j = \bigoplus_{i=1}^{K} (\mathcal{V}_i^1 \oplus \ldots \oplus \mathcal{V}_i^{R_i})$$

Because every Morse set $\mathcal{M}_i^j$ is part of a certain Morse set $\mathcal{M}_r$, it follows that the subbundle $\mathcal{V}_i^j \subseteq \mathcal{U} \times W_i$ is a subbundle of $\mathcal{V}_r$.

Now in (i), we have seen, that every $\mathcal{M}_i^j$ lies in exactly one Morse set $\mathcal{M}_{r(i,j)}$. Therefore, we get the decomposition

$$\mathcal{V}_r = \bigoplus_{(i,j)\in\mathcal{I}(\mathcal{M}_r)} \mathcal{V}_i^j$$

where $\mathcal{I}(\mathcal{M}_r)$ is an index set with $(i,j) \in \mathcal{I}(\mathcal{M}_r) \subset \{1, \ldots, K\} \times \{1, \ldots, R_i\}$. Then for every $r = 1, \ldots, R$ following equalities hold (where we abbreviate $\mathcal{I}_r := \mathcal{I}(\mathcal{M}_r)$)

$$\begin{aligned} \mathcal{V}_r &= \textstyle\bigoplus_{(i,j)\in\mathcal{I}_r} \mathcal{V}_i^j \\ &= \textstyle\bigoplus_{(i,j)\in\mathcal{I}_r} \{(u(\cdot), x_i) \in \mathcal{U} \times W_i : x_i \neq 0 \Rightarrow (u(\cdot), \mathbb{P}(x_i)) \in \mathcal{M}_i^j\} \\ &= \left\{ (u(\cdot), x) \in \mathcal{U} \times \mathbb{R}^d \ : \begin{array}{c} x = \sum_{(i,j)\in\mathcal{I}_r} x_i \text{ with } x_i \in W_i \text{ and} \\ (x_i \neq 0 \Rightarrow (u(\cdot), \mathbb{P}(x_i)) \in \mathcal{M}_i^j) \end{array} \right\} \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{M}_r &= \{(u(\cdot), \mathbb{P}x) \in \mathcal{U} \times \mathbb{P}^{d-1} : x \in \mathbb{P}^{-1}(\mathbb{P}x) \Rightarrow (u(\cdot), x) \in \mathcal{V}_r\} \\ &= \left\{ (u(\cdot), \mathbb{P}x) \in \mathcal{U} \times \mathbb{P}^{d-1} : \begin{array}{l} x \in \mathbb{P}^{-1}(\mathbb{P}x) \Rightarrow x = \sum_{(i,j)\in\mathcal{I}_r} x_i \text{ with } x_i \in W_i \\ \text{and } (x_i \neq 0 \Rightarrow (u(\cdot), \mathbb{P}(x_i)) \in \mathcal{M}_i^j) \end{array} \right\} \\ &= \left\{ (u(\cdot), \mathbb{P}x) \in \mathcal{U} \times \mathbb{P}^{d-1} : \begin{array}{l} \text{for all } (i,j) \in \mathcal{I}_r \text{ it holds} \\ \mathbb{P}x \in \mathrm{Dom}(\mathbb{P}_i) \Rightarrow (u(\cdot), \mathbb{P}_i(\mathbb{P}x)) \in \mathcal{M}_i^j \end{array} \right\} \end{aligned}$$

and the assertion (ii) is proved. □

Instead of analyzing the Morse spectrum directly, we look at the topological spectrum of the bilinear control system. For $\lambda \in \mathbb{R}$, we define new linear flows on the vector bundle $\pi : \mathcal{U} \times \mathbb{R}^d \to \mathcal{U}$ by

$$\Phi^\lambda(t, u(\cdot), x) := (\theta_t(u(\cdot)), \exp(-\lambda t)\varphi(t, x, u(\cdot))).$$

The topological spectrum of Φ is defined as

$$\Sigma_{Top}(\Phi) := \{\lambda \in \mathbb{R} : (0,0) \in \mathcal{U} \times \mathbb{R}^d \text{ is not an isolated invariant set of } \Phi_t^\lambda\} \tag{4.7}$$

Thus $\lambda \in \Sigma_{Top}$ iff there exists $(u(\cdot), x) \in \mathcal{U} \times \mathbb{R}^d \setminus (0,0)$, such that the map $t \to \Phi^\lambda(t, u(\cdot), x)$ is bounded on $\mathbb{R}$.

Similarly, we define for $\lambda \in \mathbb{R}$ linear flows Φ_i^λ on the vector bundles $\pi_i : \mathcal{U} \times W_i \to \mathcal{U}$ by restricting the flow Φ^λ:

$$\Phi_i^\lambda(t, u(\cdot), x) := (\theta_t(u(\cdot)), \exp(-\lambda t)\varphi_i(t, x, u(\cdot)))$$

and we define the topological spectrum of Φ_i as

$$\Sigma_{Top}(\Phi_i) := \{\lambda \in \mathbb{R} : (0,0) \in \mathcal{U} \times W_i \text{ is not an isolated invariant set of } \Phi_i^\lambda\} \tag{4.8}$$

With the help of the topological spectrum, we are now able to prove, that the Morse spectrum of the bilinear control system (4.1) is uniquely determined by the Morse spectra of the restricted control systems on the singular subspaces.

Theorem 4.3. Assume that the state space of the bilinear control system (4.1) is decomposable into singular subspaces $W_1, \ldots, W_K$. Then for every Morse set $\mathcal{M}_r$, the Morse spectrum of $\mathcal{M}_r$ is

$$\Sigma_{Mo}(\mathcal{M}_r) = \bigcup_{(i,j)\in\mathcal{I}(\mathcal{M}_r)} \Sigma_{Mo}(\mathcal{M}_i^j) \tag{4.9}$$

Therefore, the Morse spectrum of the bilinear control system is

$$\Sigma_{Mo} = \bigcup_{i=1}^{K} \bigcup_{j=1}^{R_i} \Sigma_{Mo}(\mathcal{M}_i^j)$$

where the union is taken over all Morse sets $\mathcal{M}_i^j$ of the induced flows $\mathbb{P}\Phi_i$ on the projective singular subbundles $\mathbb{P}\pi : \mathcal{U} \times \mathbb{P}(W_i) \to \mathcal{U}$.

Proof. From the the second part of Lemma 4.2., we have $\mathcal{M}_i^j \subset \mathcal{M}_r$ for all $(i,j) \in \mathcal{I}(\mathcal{M}_r)$. Therefore we get $\Sigma_{Mo}(\mathcal{M}_i^j) \subseteq \Sigma_{Mo}(\mathcal{M}_r)$ for all $(i,j) \in \mathcal{I}(\mathcal{M}_r)$.

Now let $\lambda \in \Sigma_{Mo}(\mathcal{M}_r)$ We use the fact, that the topological and the Morse spectrum coincide, see [**?**] (Colonius et al.,1996), Theorem 5.1. This implies

$$\Sigma_{Mo}(\mathcal{M}_r) = \Sigma_{Top}(\Phi|(\mathbb{P}\pi)^{-1}\mathcal{M}_r) = \Sigma_{Top}(\Phi|\mathcal{V}_r).$$

Therefore $\lambda \in \Sigma_{Top}(\Phi|\mathcal{V}_r)$ which means, that there exists an element $(u(\cdot), x) \in \mathcal{V}_r \setminus (0,0)$, such that the map $t \to \Phi_t^\lambda(u(\cdot), x)$ is bounded on $\mathbb{R}$. Because $\mathcal{V}_r = \bigoplus_{(i,j)\in\mathcal{I}(\mathcal{M}_r)} \mathcal{V}_i^j$ and since $x \neq 0$, it follows that there exists an $(i,j) \in \mathcal{I}(\mathcal{M})$ with $x_i := pr_i(x) \neq 0$. Here pr_i denotes the projection of $\mathbb{R}^d$ on W_i. Clearly, the map $t \to \Phi_t^\lambda(u(\cdot), x_i)$ is bounded on $\mathbb{R}$, too. Since $(u(\cdot), x_i) \in \mathcal{V}_i^j$, it follows, that $\lambda \in \Sigma_{Top}(\Phi_i|\mathcal{V}_i^j) = \Sigma_{Mo}(\mathcal{M}_i^j)$, Thus every $\lambda \in \Sigma_{Mo}(\mathcal{M}_r)$ is also an element of an appropriate $\Sigma_{Mo}(\mathcal{M}_i^j)$. □

Remark 4.4. (i) The preceding theorem shows, that the Morse spectrum of the bilinear control system (4.1) is determined by the Morse spectrum of the restricted control system on the singular subspaces. If $\lambda \in \Sigma_{Ly}$, then according to Theorem 4.1. $\lambda \in \Sigma_{Mo}(\mathcal{M}_r)$. Now Theorem 4.3. shows us, that λ must be in the Morse spectrum of some Morse set $\mathcal{M}_i^j \subset \mathcal{U} \times \mathbb{P}(W_i)$. This implies, that there is $(u(\cdot), x_i) \in \mathcal{U} \times W_i$

with $\lambda(u(\cdot), x_i) = \lambda$. This shows, that all Lyapunov exponents are attained in the subbundles $\mathcal{U} \times W_i, i = 1, \ldots, K$, and we get

$$\Sigma_{Ly} \subseteq \bigcup_{i=1}^{K} \bigcup_{j=1}^{R_i} \Sigma_{Mo}(\mathcal{M}_i^j) \tag{4.10}$$

Therefore we can restrict ourselves to these subbundles.

(ii) If the accessibility rank conditions holds on the projected singular subspaces, we can collect the results of Corollary 3.10. and Theorem 4.3. in the following way:

$$\bigcup_{i=1}^{K} \bigcup_{j=1}^{L_i} \Sigma_{Fl}(D_i^j) \subseteq \Sigma_{Ly} \subseteq \Sigma_{Mo} = \bigcup_{i=1}^{K} \bigcup_{j=1}^{R_j} \Sigma_{Mo}(\mathcal{M}_i^j) \tag{4.11}$$

where the first union is taken over all main control sets D_i^j on the projected singular subspaces $\mathbb{P}(W_i)$, and the second union over all Morse sets $\mathcal{M}_i^j$ on the projected subbundles $\mathcal{U} \times \mathbb{P}(W_i)$.

5. The Lyapunov spectrum

In this section we give conditions under which the inequalitites in (4.11) become equalities. Under equality, we can determine the Lyapunov spectrum by considering only the restricted control systems. If the dimension of the singular subspaces is low enough, we can take numerical programs to compute the Lyapunov spectrum of the *whole* system by computing the Lyapunov spectra of the restricted systems.

In the paper of Colonius and Kliemann (1996) [**?**], several conditions are specified under which the Morse, the Lyapunov and the closure of the Floquet spectrum coincide, i.e. under which the equality

$$\bigcup_{i=1}^{L} \mathrm{cl}\Sigma_{Fl}(D_i) = \Sigma_{Ly} = \Sigma_{Mo} \tag{5.1}$$

holds. In Corollary 4.8 in [**?**], it was shown, that this equality holds if the control system fulfills the accessibility rank condition (ARC) and has only one control set with nonvoid interior on the projective space $\mathbb{P}^{d-1}$. Furthermore it was shown in Corollay 4.10 in [**?**], that if the state space of the control system is two-dimensional and the (ARC) is fulfilled, then equality (5.1) holds, too. A third criterion is given by varying the control range U. For explaining this criterion, we have to introduce some notations.

For the compact and convex control range $\mathrm{U} \subset \mathbb{R}^m$ with $0 \in \mathrm{intU}$ denote

$$\mathrm{U}^\rho := \rho\mathrm{U} = \{\rho \cdot u : u \in \mathrm{U}\} \text{ for } \rho \geq 0. \tag{5.2}$$

For $\rho = \infty$ we define $\mathcal{U}^\infty := \bigcup_{\rho>0} \mathcal{U}^\rho = \mathrm{L}^\infty(\mathbb{R}, \mathbb{R}^m)$. All quantitites defined in Sections 2 -4 will be written with a superscript ρ to indicate their dependence on the control range U^ρ for $0 \leq \rho \leq \infty$. Like in Section 3, we define the nondegeneracy condition (ARC$^\rho$) by

$$\dim\mathcal{LA}\{h(\cdot, u) : u \in \mathrm{U}^\rho\}(x) = d - 1 \text{ for all } x \in \mathbb{P}^{d-1}, \text{ all } \rho > 0. \tag{ARC$^\rho$}$$

Since $0 \in \text{intU}$, we have that if (ARC^ρ) holds for some $\rho > 0$, then it holds for all $\rho > 0$. Furthermore, we introduce the $\rho - \rho'$ inner pair condition (IPC).

$$\begin{array}{l}\text{For all } 0 \leq \rho \leq \rho' \text{ and all } (u(\cdot), x) \in \mathcal{U}^\rho \times \mathbb{P}^{d-1} \text{ there exists } T > 0 \\ \text{and } S > 0 \text{ such that } \mathbb{P}\varphi(T, x, u(\cdot)) \in \text{int}\mathcal{O}^{\rho'+}_{\leq T+S}(x)\end{array} \tag{IPC}$$

If the projected control system (4.2) fulfills the condition (ARC^ρ) and (IPC) then Corollary 5.6 in [**?**] says, that for all $\rho \geq 0$ (except at most countably many points) the following equation holds

$$\bigcup_{i=1}^{L} \text{cl}\Sigma_{Fl}(D_i^\rho) = \Sigma_{Ly}^\rho = \Sigma_{Mo}^\rho.$$

Here D_i^ρ denotes the main control sets of the bilinear control system (4.1^ρ) with control range U^ρ. The equality holds at those ρ for which the spectra depends continuously on ρ, cp. Corollary 5.6 in [**?**].

This results supposes that the accessibility rank condition holds for the projected control system. But if the state space of the bilinear control system is decomposable into singular subspaces $W_1, \ldots, W_K$, the accessibility rank condition is not fulfilled, as we have shown in Corollary 3.3. Therefore instead of demanding that the whole system possesses the (ARC), we impose, that each of the control systems on the projected singular subspaces $\mathbb{P}(W_i)$ fulfills the (ARC). By doing this, we can adopt the results on the singular subspaces.

Theorem 5.1. Assume, that the state space of the bilinear control system (4.1) is decomposable into a direct sum of singular subspaces $W_1, \ldots, W_K$. Assume, that each of the systems on the singular subspaces W_i fulfill at least one of the following properties

(i) the state space W_i is one-dimensional

(ii) the state space W_i is two-dimensional and the accessibility rank condition (ARC) holds on the projected control system on $\mathbb{P}(W_i)$.

(iii) the accessibility rank condition (ARC) holds on the projected control system on $\mathbb{P}(W_i)$ and only one main control set $D_i \subseteq \mathbb{P}(W_i)$ exists.

(iv) the accessibility rank condition (ARC^ρ) and the $\rho - \rho'$ -inner pair condition (IPC) holds on the projected control system on $\mathbb{P}(W_i)$ and $\rho = 1$ is a continuity point.

Then following equalities hold

$$\bigcup_{i=i}^{K} \bigcup_{j=i}^{L_i} \text{cl}\Sigma_{Fl}(D_i^j) = \Sigma_{Ly} = \Sigma_{Mo} = \bigcup_{i=1}^{K} \bigcup_{j=1}^{R_i} \Sigma_{Mo}(\mathcal{M}_i^j) \tag{5.3}$$

where the first union is taken over all main control sets D_i^j on the projected singular subspaces $\mathbb{P}(W_i)$, and the second union over all Morse sets $\mathcal{M}_i^j$ on the projected subbundles $\mathcal{U} \times \mathbb{P}(W_i)$.

Proof. We show, that for every $i = 1, \dots, K$ the following equations holds.

$$\bigcup_{j=1}^{L_i} \mathrm{cl}\Sigma_{Fl}(D_i^j) = \Sigma_{Ly}(W_i) = \bigcup_{j=1}^{R_j} \Sigma_{Mo}(\mathcal{M}_i^j) \tag{5.4}$$

Here $\Sigma_{Ly}(W_i)$ denotes the Lyapunov spectrum of the restricted control system on W_i. Because of (4.10), this will show the assumption.

If the singular subspace W_i is one-dimensional, the accessibility rank condition is satisfied, because $\dim\mathbb{P}(W_i) = 0$. The restricted control system on $\mathbb{P}(W_i)$ has exactly one main control set, namely $\mathbb{P}(W_i)$ itself. Thus Corollary 4.8 in [?] shows, that the equation (5.4) holds. If the singular subspace W_i is two-dimensional and the (ARC) is fulfilled, equation 5.4 follows from Corollary 4.10 in [?]. If there exists only one main control set, then again Corollary 4.8 in [?] proves equality. Finally, if we are in case (iv), the equation follows from Corollary 5.6 in [?]. □

In the next section we give an example, where every singular subspace is two-dimensional and we are in the situation (ii) of Theorem 5.1.

6. An Example

For the computation of the Lyapunov spectrum of bilinear control systems, one depends on numerical algorithms. E.g. in [?] (Grüne, 1996) the extremal Lyapunov exponents for bilinear control systems with constrained control values are computed numerically by solving discounted optimal control problems. All these algorithms have in common, that the memory requirements and the computation time grow very fast as the dimension of the state space of the systems increase. This is the well known *curse of dimension* problem in numerical mathematics. As mentioned earlier, the aim behind looking at bilinear control systen with singular subspaces is to reduce the dimension of the original problem. Instead of computing the Lyapunov spectrum of one high-dimensional control system, one computes the Lyapunov spectrum of several lower-dimensional systems, which can be done with smaller costs of memory and time. In this section we look at four coupled two-dimensional oscillators, which results in an eight-dimensional bilinear control system. But the coupling we investigate here inherits some symmetry properties, which make a decoupling of the eight-dimensional system into four different two-dimensional bilinear control systems possible. This allows us to determine the Lyapunov spectrum analytically, using the results of the previous section. The example is based on the paper of Aston and Dellnitz (1994) [?] Chapter 4, where the symmetry properties of general linear coupling arrangements were investigated.

We look at the bilinear control system

$$\begin{array}{ll} \dot{x}(t) = A_0 x(t) + u_1(t) A_1 x(t) + u_2(t) A_2 x(t), \qquad t \in \mathbb{R}, \qquad x(0) = x_0 \in \mathbb{R}^d, \\ u(\cdot) \in \mathcal{U}_{loc} := \{u(\cdot) : \mathbb{R} \to \mathrm{U} : u(\cdot) \text{ locally integrable}\}, \end{array} \tag{6.1}$$

where $\mathrm{U} = [-\frac{1}{4}, \frac{1}{4}] \times [-\frac{1}{2}, \frac{1}{2}]$ and $A_j \in \mathrm{Mat}(2, \mathbb{R})$ with

$$A_0 = \begin{pmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{2} \end{pmatrix}, \; A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now we couple four of these oscillators in the following way together:

$$\dot{x}_i = A_0 x(t) + u_1(t) A_1 x_i(t) + u_2(t) A_2 x_i(t) + \sum_{j=1}^{4} C_{i,j}(a,b) x_j(t) \tag{6.2}$$

with $i = 1, \ldots, 4$ where $C_{i,j}(a,b)$ denotes the components of the coupling matrix $C(a,b) \in \mathrm{Mat}(8, \mathbb{R})$ which depends on the real parameters a and b. We investigate the special case, where the coupling matrix has the form

$$C(a,b) = \begin{pmatrix} -(a+b)I & aI & 0 & bI \\ aI & -(a+b)I & bI & 0 \\ 0 & bI & -(a+b)I & aI \\ bI & 0 & aI & -(a+b)I \end{pmatrix} \tag{6.3}$$

and $I \in \mathrm{Mat}(2, \mathbb{R})$ is the unit matrix. Here the two parameters a and b describe the strength of the coupling between the four oscillators. This system possesses a certain symmetry structure, which enables us to decouple the system into four bilinear control systems. For a discussion of the symmetry properties for more general $C(a,b)$ and the techniques used for the decomposition compare [**?**] (Grünvogel, 1996). In our situation we get by an orthogonal change of coordinates the four decoupled control systems

$$\dot{x}_i = \tilde{A}_0^i x_i(t) + u_1(t) A_1 x_i(t) + u_2(t) A_2 x(t) \tag{6.4}$$

with $i = 1 \ldots 4$ and

$$\tilde{A}_0^0 = A_0,\ \tilde{A}_0^1 = A_0 - 2aI,\ \tilde{A}_0^2 = A_0 - 2bI \text{ and } \tilde{A}_0^3 = A_0 - 2(a+b)I$$

This means, that the first control system ($i = 1$) coincides with the original control system (6.1), and the others are just pertubations of this control system.

In order to apply the result of Theorem 5.1. (ii), we have to check that the projected contol systems on $\mathbb{P}^1$ fulfills the pre-conditions. It is easy to see, that on each projected control system the accessibility rank condition holds. Next we show, that for every $i = 1, \ldots, 4$ the associated control system on the projective singular subspace $\mathbb{P}^1$ has two main control sets and one chain control set. This enables us to compute the Floquet spectra of the main contol sets by calculating the eigenvalues of the constant matrixes of (6.4). In order to determine the main control sets we parametrize the projective space $\mathbb{P}^1$ (or in fact $\mathbb{S}^1$) via the angle as $\mathbb{P}^1 = \{\theta : \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$.

For all four control systems, the projected control system has the form

$$\dot{\theta} = -(\frac{1}{4} + u_1) \sin^2\theta + (\frac{1}{4} + u_1) \cos^2\theta + (\frac{1}{2} + u_2) \sin\theta \cos\theta$$

That means, that on the three restricted control systems $i = 2 \ldots 4$ on $\mathbb{P}^1$, we have the same global behaviour as on the unperturbed control system for $i = 1$. In [**?**] the unpertubed control system was investigated. The result is, that for every $i = 1, \ldots, 4$ we get two main control sets, the variant control set $D_i^1 = (-\frac{\pi}{2}, 0)$, and the invariant control set $D_i^2 = [\frac{\pi}{4}, \frac{\pi}{2}]$. The unique chain control sets are in each case $\mathbb{P}^1$. Therefore, we can compute the Floquet spectrum of the restricted control systems, from the eigenvalues of the constant matrices of (6.4). The Floquet spectrum for the eight main control sets are

$$\begin{aligned}
\Sigma_{Fl}(D_1^1) &:= \left(\tfrac{1}{2}, 1\right), & \Sigma_{Fl}(D_1^2) &:= \left(1, \tfrac{3}{2} + \tfrac{1}{2}\sqrt{2}\right) \\
\Sigma_{Fl}(D_2^1) &:= \left(\tfrac{1}{2} - 2a, 1 - 2a\right), & \Sigma_{Fl}(D_2^2) &:= \left(1 - 2a, \tfrac{3}{2} + \tfrac{1}{2}\sqrt{2} - 2a\right) \\
\Sigma_{Fl}(D_3^1) &:= \left(\tfrac{1}{2} - 2b, 1 - 2b\right), & \Sigma_{Fl}(D_3^2) &:= \left(1 - 2b, \tfrac{3}{2} + \tfrac{1}{2}\sqrt{2} - 2b\right) \\
\Sigma_{Fl}(D_4^1) &:= \left(\tfrac{1}{2} - 2(a+b), 1 - 2(a+b)\right), & \Sigma_{Fl}(D_4^2) &:= \left(1 - 2(a+b), \tfrac{3}{2} + \tfrac{1}{2}\sqrt{2} - 2(a+b)\right)
\end{aligned}$$

Hence by (5.4) we get the following Lyapunov spectra on the singular subspaces:

$$\begin{aligned}
\Sigma_{Ly}^1 &= \left[\tfrac{1}{2}, \tfrac{3}{2} + \tfrac{1}{2}\sqrt{2}\right] & \Sigma_{Ly}^2 &= \left[\tfrac{1}{2} - 2a, \tfrac{3}{2} + \tfrac{1}{2}\sqrt{2} - 2a\right] \\
\Sigma_{Ly}^3 &= \left[\tfrac{1}{2} - 2b, \tfrac{3}{2} + \tfrac{1}{2}\sqrt{2} - 2b\right] & \Sigma_{Ly}^4 &= \left[\tfrac{1}{2} - 2(a+b), \tfrac{3}{2} + \tfrac{1}{2}\sqrt{2} - 2(a+b)\right]
\end{aligned}$$

Here Σ_{Ly}^i denotes the Lyapunov spectrum of the bilinear control system on the singular subspaces W_i, depending on the real parameters a and b. Now the Lyapunov spectrum can be computed with help of Theorem 5.1. as

$$\Sigma_{Ly} = \Sigma_{Ly}^1 \cup \Sigma_{Ly}^2(a) \cup \Sigma_{Ly}^3(b) \cup \Sigma_{Ly}^4(a,b) \tag{6.5}$$

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