Lyapunov Exponents and Control Sets near Singular Points

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Abstract

We consider control affine systems of the form

$$\dot{x} = f_0(x) + \textstyle\sum_{i=1}^{m} u_i(t) f_i(x),$$
$$u \in \mathcal{U} = \{u : \mathbb{R} \to U : \text{loc. integrable}\}$$

on $\mathbb{R}^d$ with compact control range U and a singular point $x^* \in \mathbb{R}^d$, i.e. $f_i(x^*) = 0, i = 0, \dots, m$. Control sets are maximal subsets of $\mathbb{R}^d$ with nonvoid interior where the system is approximately controllable. We suppose that there is a periodic control functions u^h such that linearized the system has positive and negative Lyapunov exponents for u^h and a periodic control function u^s such that the linearized system has only negative Lyapunov exponents for u^s. Under an inner pair condition we show the existence of control sets near the singular point, by using local stable and unstable manifolds of the system corresponding to u^h and its asymptotic phase.

Key words: control set, Lyapunov exponent, invariant manifold

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1 Introduction

The analysis of local controllability properties is still an important item in control theory and there are many different attempts of attacking this problem. A relatively new one is to consider the control system $\dot{x} = f(x, u)$ as a dynamical system, where the set of control functions $\mathcal{U}$ is part of the state space of this dynamical system (see [1] for more details). We consider here control systems which have a singular point, i.e. which have an equilibrium which remains fixed under every control. If one considers the control range as a parameter, then numerical experiments have shown, that there may be a kind of bifurcation of control sets near the singular point.

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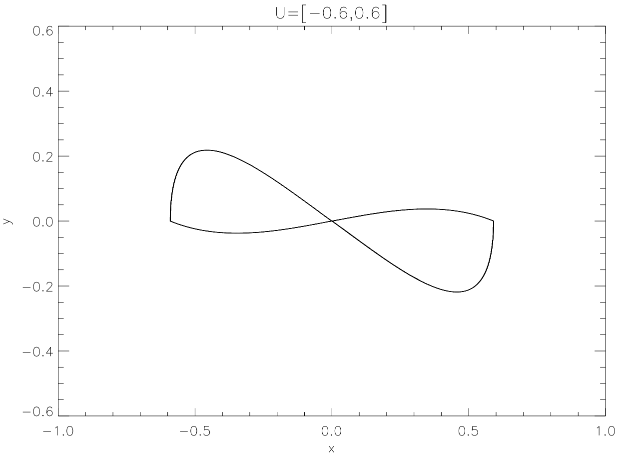


Fig. 1. Control sets for the Duffing-van der Pol equation.

Example 1 *Consider the perturbed Duffing-van der Pol oscillator*

$$\begin{cases} \dot{x} = y \\ \dot{y} = (\alpha + u(t))x - \beta y - x^3 - x^2 y \end{cases} \tag{1}$$

$$u \in \mathcal{U}_\rho := \{u : \mathbb{R} \to \mathbb{R} : u(t) \in U_\rho \textit{ for a.a. } t \in \mathbb{R}, \textit{ locally integrable}\}$$

with $U_\rho = [-\rho, \rho] \subset \mathbb{R}$. *Then, for* $0 < \rho < \frac{1}{4}$, *we find numerically that there is no control set with nonvoid interior. For* $\rho > \frac{1}{4}$ *we observe numerically two control sets with nonvoid interior, such that the singular point lies in the closure of the both control sets, cf. Figure 1 for* $\rho = 0.6$. *Here the two bubbles emerging from the origin are control sets.*

The key to understand this behavior is to consider the bilinear control system which we get, if we linearize the nonlinear control system at the singular point. In the example above for $0 < \rho < \frac{1}{4}$ the bilinear control system has only negative Lyapunov exponents for every control function. For systems of this kind there are no control sets locally around the singular point (cf. [2]). For $\rho > \frac{1}{4}$ we can find a periodic control function u^h with positive and negative Lyapunov exponents. Thus the idea is to use this property to show the existence of control sets.

In Section 2 we introduce the notion of control sets and control flow. Then we state a general existence criterion for control sets in terms of ω-limit sets of the control flow. The idea is to construct a control function $u \in \mathcal{U}$ by two periodic control functions u^h and u^s and to find a starting point in state space such that the trajectory is bounded and has a certain limit behavior. The idea is explained in Section 3. The linearized system corresponding to u^h has positive and negative Lyapunov exponents and that to u^s only negative ones. For the nonlinear system of u^h we introduce in Section 4 the corresponding local stable and unstable fibre bundles which are generalizations of stable and unstable manifolds for time varying systems. In Section 5 we gather some technical prerequisites we need for the following section. Then in Section 6 we construct u and characterize the corresponding ω-limit set in Section 7. Finally in Section 8 we gather all results and show the existence theorem for control sets near the singular point.

2 Control Sets and Control Flow

We consider the control affine system on $\mathbb{R}^d$

$$\begin{aligned} &\dot{x} = f_0(x) + \textstyle\sum_{i=1}^{m} u_i(t) f_i(x) \\ &u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\} \end{aligned} \tag{2}$$

where U is a compact and convex subset of $\mathbb{R}^m$. We assume that $f_0, \ldots, f_m$ are C^2 vector fields on $\mathbb{R}^d$. Furthermore we suppose that for all $(u, x) \in \mathcal{U} \times \mathbb{R}^d$ the equation (2) has a unique solution $\varphi(t, x, u), t \in \mathbb{R}$, with $\varphi(0, x, u) = x$.

We suppose, that the system (2) has the singular point $x^* = 0 \in \mathbb{R}^d$, i.e.

$$f_i(x^*) = 0 \text{ for } i = 0, \ldots, m$$

This means, that $\varphi(t, x^*, u) = x^*$ for all $u \in \mathcal{U}, t \in \mathbb{R}$.

The set of points, which are reachable from a given point $x \in M$ is

$$\mathcal{O}^+(x) := \left\{ y \in \mathbb{R}^d : \text{there is } t \geq 0 \text{ and } u \in \mathcal{U} \text{ with } y = \varphi(t, x, u) \right\}.$$

Definition 2 *A set $D \subset \mathbb{R}^d$ with $intD \neq \emptyset$ is called* control set *of the system (2) if for all $x \in D$ one has $D \subset cl\mathcal{O}^+(x)$. Furthermore D has to be maximal with these properties, that is, if $D' \supset D$ satisfies the conditions above, then $D' = D$.*

This means we have *approximate controllability* in D. For every $x, y \in D$ and every neighborhood N of y there exists a control function $u \in \mathcal{U}$ and a time $t \geq 0$ such that $\varphi(t, x, u) \in N$. If the control set has nonvoid interior, then under local accessibility, we even have complete controllability in the interior of the control set.

Control sets can be characterized by the ω-limit sets of trajectories. For this purpose we first have to introduce a dynamical system on $\mathcal{U}$. The *shift* $\theta : \mathbb{R} \times$

$\mathcal{U} \to \mathcal{U}$ is defined by $\theta_t(u) := \theta(t,u) := u(t+\cdot)$. We call the pair $(\mathcal{U},\theta)$ the *shift space*. If we endow $\mathcal{U}$ with the weak* topology of $L_\infty(\mathbb{R},\mathbb{R}^m) = (L_1(\mathbb{R},\mathbb{R}^m))^*$, then it is a compact complete separable metric space and $(\mathcal{U},\theta)$ is a continuous dynamical system (cf.Lemma 4.2.1 in [1]).

By combining the shift with the solution mapping φ we obtain the *control flow* defined by

$$\begin{aligned} \Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d \to &\quad \mathcal{U} \times \mathbb{R}^d \\ (t,u,p) &\mapsto (\theta_t(u), \varphi(t,p,u)). \end{aligned}$$

Φ defines a continuous dynamical system on $\mathcal{U} \times \mathbb{R}^d$ (cf. Lemma 4.3.2 in [1]). The $\mathbb{R}^d$-component satisfies the cocycle property $\varphi(t+\tau,x,u) = \varphi(t,\varphi(\tau,x,u),\theta_\tau(u))$. Hence Φ is a skew-product flow. Because $(\mathcal{U}\times\mathbb{R}^d,\Phi)$ is a continuous dynamical system, we can define the ω-limit set of a pair $(u,x) \in \mathcal{U} \times \mathbb{R}^d$:

$$\omega(u,x) := \left\{ (v,y) \in \mathcal{U} \times \mathbb{R}^d : \begin{array}{l} \text{there is a sequence } (t_k)_{k\in\mathbb{N}} \in \mathbb{R} \\ \text{with } t_k \to \infty \text{ for } k \to \infty \text{ and} \\ \lim_{k\to\infty} \Phi_{t_k}(u,x) = (v,y) \end{array} \right\}.$$

If for a given $(u,x) \in \mathcal{U} \times \mathbb{R}^d$ the set $\{\varphi(t,x,u) : t \geq 0\}$ is bounded, then the ω-limit set $\omega(u,x)$ is nonempty, invariant and compact.

In order to describe the connection between ω-limit sets and control sets, we have to introduce the following notions.

Definition 3 *A pair $(u,x) \in \mathcal{U} \times \mathbb{R}^d$ is called* inner pair, *if there exists a $T > 0$ such that $\varphi(T,x,u) \in int\mathcal{O}^+(x)$. The pair (u,x) is called* strong inner pair *if for all $t > 0$ we have $\varphi(t,x,u) \in int\mathcal{O}^+(x)$.*

Remark 4 *For constant control u the following criterion allows to check, if (u,x) is a strong inner pair: Write $f = f_0 + \sum_{i=1}^m u_i f_i$ and assume that there is a $T > 0$ such that at $y = \varphi(T,x,u)$*

$$span\{f(y), ad_f^k f_i(y), i = 1,\dots,m\} = \mathbb{R}^d.$$

Here for two vector fields X,Y ad is defined as $ad_X Y := [X,Y]$ (the Lie bracket of X and Y) and $ad_X^k Y := ad_X(ad_X^{k-1} Y)$ for $k \geq 2$. The above condition implies a local controllability along the trajectory. It also follows that each $(v,y) \in \mathcal{U} \times \mathbb{R}^d$, where v restricted to $[0,T]$ is in a neighborhood of $u \in L_\infty(0,T,\mathbb{R}^m)$ and y in an neighborhood around x, is an inner pair. If $0 \in intU$, the rank condition above is satisfied, and U is small enough, then *it is shown in [1] Corollary 4.7.6 that the inner pair condition holds for all u with values uniformly in the interior of U.*

Remark 5 *Suppose, that the system (2) is equivalent to the dth order equation with additive control*

$$\begin{aligned}&y^{(d)}(t)+a_{n-1}y^{(d-1)}+\ldots+a_1y(t)+a_0=b(y^{(d)}(t),\ldots,y(t))u(t),\\&u\in\mathcal{U}\end{aligned}$$

where $a_i \in \mathbb{R}$ for $i=0,\ldots,d-1, b:\mathbb{R}^d\to\mathbb{R}$ is a bounded C^1-function with $|b|>\gamma>0$, and $U=[\alpha,\beta]$ is a closed interval in $\mathbb{R}$ containing the origin. Then by Proposition 5 in [3] a pair $(u,x)\in\mathcal{U}\times\mathbb{R}^d$ is a strong inner pair if $x\neq 0$ and there is an $\varepsilon>0$ such that $u(t)\in[\alpha+\varepsilon,\beta-\varepsilon]$ for all $t\in\mathbb{R}$. In the proof the equation is solved for $u(t)$, when a neighborhood of the final point of equation (2) is prescribed. Similar arguments also apply to the (nonlinear) Duffing-van der Pol oscillator in Example 1.

The following proposition is crucial for the construction in Section 6 and the existence Theorem 20. It shows that the ω-limit set of a trajectory may have a nonvoid intersection with a control set. The projection $\pi_{\mathbb{R}^d}$ of $\mathcal{U}\times\mathbb{R}^d$ onto $\mathbb{R}^d$ is defined by $\pi_{\mathbb{R}^d}(u,p):=p$.

Proposition 6 *Consider the nonlinear control system (2). Assume that $\mathbb{R}^d\setminus\{0\}$ and $\{0\}$ are maximal integral manifolds. Let $x\in\mathbb{R}^d\setminus\{0\}$ and $u\in\mathcal{U}$, such that $\{\varphi(t,x,u):t\geq 0\}$ is bounded. Suppose further that there is a compactum $\mathcal{K}\subset\mathcal{U}\times(\mathbb{R}^d\setminus\{0\})$ such that the following properties are satisfied.*

(1) $\mathcal{K}\cap\omega(u,x)\neq\emptyset$.
(2) For all $(v,y)\in\mathcal{K}$ there is a $t>0$ such that $\varphi(t,y,v)\in int\mathcal{O}^+(y)$ (inner pair condition).
(3) There is a $s^*>0$ such that for all $(v,y)\in\Phi_{-s^*}\mathcal{K}$ and all $t>0$ we have $\varphi(t,y,v)\in int\mathcal{O}^+(y)$ (strong inner pair condition).

Then there exists a control set $D\subset\mathbb{R}^d$ with

$$\pi_{\mathbb{R}^d}(\mathcal{K}\cap\omega(u,x))\subset intD. \tag{3}$$

PROOF. Since $\mathbb{R}^d\setminus\{0\}$ and $\{0\}$ are maximal integral manifolds, the system restricted to $\mathbb{R}^d\setminus\{0\}$ is locally accessible. Let $(v,y),(w,z)\in\mathcal{K}\cap\omega(u,x)$.We show, that $z\in\mathrm{int}\mathcal{O}^+_{\leq\tau}(y)$ for a $\tau>0$.

Because of $(v,y) \in \mathcal{K}$ for $T_0 > 0$ there is by *(2)* an $\varepsilon_0, S_0 > 0$ with

$$B_{\varepsilon_0}(\varphi(T_0,y,v)) \subset \operatorname{int}\mathcal{O}^+_{\leq T_0+S_0}(y) \tag{4}$$

Because $(v,y) \in \omega(u,x)$ there is a $t_0 > 0$ with

$$\varphi(t_0,x,u) \in B_{\varepsilon_0}(\varphi(T_0,y,v)) \subset \operatorname{int}\mathcal{O}^+_{\leq T_0+S_0}(y) \tag{5}$$

By *(3)* and Remark 4.5.6 in [1] we get with $T_1 := s^*$ a neighborhood $\mathcal{N}$ of $\Phi_{-T_1}\mathcal{K}$ and $\varepsilon_1, S_1 > 0$ with

$$B_{\varepsilon_1}(\varphi(T_1,p,a)) \subset \operatorname{int}\mathcal{O}^+_{\leq T_1+S_1}(p) \text{ for all } (a,p) \in \mathcal{N}.$$

Note that because of $(\theta_{-T_1}w, \varphi(-T_1,z,w)) \in \Phi_{-T_1}(\mathcal{K})$ it follows that $(\theta_{-T_1}w, \varphi(-T_1,z,w)) \in \mathcal{N}$. There are open neighborhoods $V \subset (\mathbb{R}^d \setminus \{0\})$ of $\varphi(-T_1,z,w)$ and $W \subset \mathcal{U}$ of $\theta_{-T_1}w$ with $W \times V \subset \mathcal{N}$. We can choose V small enough, such that

$$\varphi(T_1,V,\theta_{-T_1}w) \subset B_{\frac{\varepsilon_1}{2}}(z). \tag{6}$$

Since $(w,z) \in \omega(u,x)$ and because of invariance of $\omega(u,x)$ it follows that $(\theta_{-T_1}w, \varphi(-T_1,z,w)) \in \omega(u,x)$. Thus there is a time $t_1 > t_0$ such that $\varphi(t_1,x,u) \in V$, and we get $(\theta_{-T_1}w, \varphi(t_1,x,u)) \in W \times V \subset \mathcal{N}$. This means, that

$$B_{\varepsilon_1}(\varphi(T_1,\varphi(t_1,x,u),\theta_{-T_1}w)) \subset \operatorname{int}\mathcal{O}^+_{\leq T_1+S_1}(\varphi(t_1,x,u)).$$

Now define the control function $\tilde{u} : \mathbb{R} \to U$ by

$$\tilde{u}(t) := \begin{cases} u(t) & \text{for } t < t_1, \\ w(t - T_1 - t_1) & \text{for } t \geq t_1. \end{cases}$$

Together with (5) and (6) we get

$$\begin{aligned} z &\in B_{\varepsilon_1}(\varphi(t_1+T,x,\tilde{u})) \\ &\subset \operatorname{int}\mathcal{O}_{\leq T_0+S_0+T_1+S_1+t_1-t_0}(y) \end{aligned}$$

Thus it follows, there exists a $\tau > 0$ with $z \in \operatorname{int}\mathcal{O}^+_{\leq\tau}(y)$ and $y \in \operatorname{int}\mathcal{O}^+_{\leq\tau}(z)$. Now there exists a neighborhood V_z of z with $V_z \subset \operatorname{int}\mathcal{O}^+(y)$. By local accessibility on J the set $V := \operatorname{int}\mathcal{O}^-(z) \cap V_z$ is nonvoid. Every $p \in V$ can be reached from y and hence from z and therefore from every point in $\operatorname{int}\mathcal{O}^-(z)$.

Thus $V \subset D$ for some control set D. Finally, since $z \in D$ can be reached in finite time from $\mathrm{int}V \subset \mathrm{int}D$, it follows that $z \in \mathrm{int}D$. Thus we have shown $\pi_{\mathbb{R}^d}(\mathcal{K} \cap \omega(u,x)) \subset \mathrm{int}D$. $\Box$

3 Basic Idea

If we linearize the nonlinear control system (2) at the singular point $x^* = 0$ we get the bilinear control system on $\mathbb{R}^d$:

$$\begin{aligned} &\dot{x} = A_0 x + \textstyle\sum_{i=1}^m u_i(t) A_i x \\ &u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\} \end{aligned} \tag{7}$$

where $A_i := \frac{\partial f_i}{\partial x}\Big|_{x=0}$. For $u \in \mathcal{U}$ we denote the fundamental solution of (7) by $\eta(t,\tau,u)$, with $\eta(\tau,\tau,u)x = x$, where $\tau, t \in \mathbb{R}, x \in \mathbb{R}^d$.

To analyze the stability behavior of the nonautonomous linear system we have to consider its Lyapunov exponents. The Lyapunov exponent for $(u,x) \in \mathcal{U} \times \mathbb{R}^d, x \neq 0$ is defined by $\lambda(u,x) = \limsup_{t\to\infty} \frac{1}{t} \ln \|\eta(t,0,u)x\|$. For a given $u \in \mathcal{U}$ the set $\lambda(u) := \{\lambda(u,x) : x \in \mathbb{R}^d, x \neq 0\}$ consists of at most d elements (cf. Hahn [4]).

For the rest of this article, we suppose, that the following conditions are satisfied.

Condition 1 *We assume, that there exist two periodic control functions $u^h, u^s \in \mathcal{U}$ with the following properties:*

(a) The control function u^h has period $\Theta \geq 0$ and the Lyapunov exponents $\lambda(u^h) = \{\lambda_1^h, \ldots, \lambda_d^h\}$ of the corresponding linear system $\dot{x} = A_0 x + \sum_{i=1}^m u_i^h(t) A_i x$ have the property

$$\lambda_1^h \geq \ldots \geq \lambda_n^h > 0 > \lambda_{n+1}^h \geq \ldots \geq \lambda_d^h \tag{8}$$

for a $n \in \{1, \ldots, d-1\}$.

(b) For the control function u^s the corresponding linear system $\dot{x} = A_0 x + \sum_{i=1}^m u_i^s(t) A_i x$ has the Lyapunov exponents $\lambda(u^s) = \{\lambda_1^s, \ldots, \lambda_d^s\}$ with the property

$$0 > \lambda_1^s \geq \lambda_2^s \geq \ldots \geq \lambda_d^s. \tag{9}$$

We call u^h the hyperbolic control function *and u^s the* stable control function.

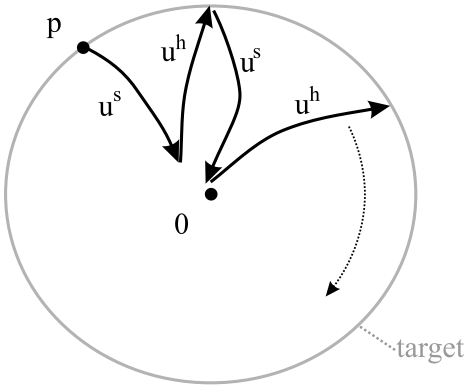


Fig. 2. Basic construction idea of the control function u.

Remark 7 *The set* $\Sigma_{Ly} = \{\lambda(u,x) : (u,x) \in \mathcal{U}\times\mathbb{R}^d, x \neq 0\}$ *is called the* Lyapunov spectrum *and the set* $\Sigma_{Fl} = \{\lambda(u,x) : (u,x) \in \mathcal{U}\times\mathbb{R}^d, x \neq 0, u$ *is piecewise constant and* τ*-periodic for some* $\tau \geq 0$ *with* $\eta(\tau,0,u)x = \alpha\eta(0,0,u)x$ *for some* $\alpha > 0\}$ *the* Floquet spectrum *of the bilinear control system. Under appropriate assumptions, the set* Σ_{Ly} *consists of compact intervals and* $cl\Sigma_{Fl} = \Sigma_{Ly}$ *(cf. Colonius and Kliemann [1]). If none if these intervals contains* 0 *then we can not find two controls* u^h *and* u^s *with the properties above. If* 0 *is in the interior of one of these intervals, then there may be two controls* u^h *and* u^s *with the properties above; and because of* $cl\Sigma_{Fl} = \Sigma_{Ly}$ *we can even find in this case piecewise constant controls with this property. Cf. [2] for a discussion of this criterion.*

To obtain the existence of control sets with nonvoid interior under the Condition 1, the basic idea is to apply Proposition 6. Thus we need a pair $(u,p) \in \mathcal{U}\times\mathbb{R}^d$ such that $\{\varphi(t,p,u) : t \geq 0\}$ is bounded. We will construct such a control function as follows. First we choose a starting point $p \in \mathbb{R}^d$, close enough to the singular point 0, such that we can characterize the qualitative behavior of the ordinary differential equations corresponding to u^h and u^s. The system corresponding to u^s is locally asymptotic stable. For u^h we get local stable and unstable fibre bundles, which means that if we start with a point near the singular point 0, the solutions are driven away from 0. The idea is to construct u by switching between u^h and u^s (cf. Figure 2).

We steer some time towards the origin with u^s. Then we switch to u^h and get driven away from the singular point 0, until we reach some set, the so called target set. Then we steer towards the origin with u^s and so forth. By choosing the appropriate switching times, we can achieve, that $\pi_{\mathbb{R}^d}\omega(u,p) \neq \{0\}$.

4 The Hyperbolic System

In this section we will have a closer look at the *hyperbolic* system

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i^h(t) f_i(x). \tag{10}$$

Because (10) is a nonautonomous system, the classical theory of stable and unstable manifolds does not apply. We here follow Aulbach and Wanner [5] (see also Siegmund [6]) who introduced the corresponding stable and unstable fibre bundles. The existence theorem for these fibre bundles demand certain quantitative properties. Because the given system (10) does in general not fulfil these criterions, we apply a cut-off technique to get an appropriate system. Thus the results will be local.

Because the vector fields f_i are C^2-vector fields, we can write

$$f_i(x) = A_i x + F_i(x)$$

where $F_i(x)$ is a continuously differentiable vector field, with $\left.\frac{\partial F_i}{\partial x}\right|_{x=0} = 0$. Thus system (10) can be written in the form

$$\dot{x} = A(t)x + F(t,x) \tag{11}$$

with

$$\begin{aligned} A(t)x &:= \quad A_0 x + \textstyle\sum_{i=1}^{m} u_i^h(t) A_i x \\ F(t,x) &:= F_0(x) + \textstyle\sum_{i=1}^{m} u_i^h(t) F_i(x). \end{aligned}$$

By Floquet theory (cf. Sansonse and Conti [7]) we get subspaces X, Y of $\mathbb{R}^d$ where X corresponds to $\lambda_1^h, \ldots, \lambda_n^h$ and Y corresponds to $\lambda_{n+1}^h, \ldots, \lambda_d^h$ and a transformation

$$\mathcal{F} : \mathbb{R} \times \mathbb{R}^d \to X \times Y$$

which is 2Θ-periodic in the first component such that the original system (10) gets transformed into following differential equation on $X \times Y$

$$\begin{aligned} \dot{x} &= A^+ x + F^+(t,x,y) \\ \dot{y} &= A^- y + F^-(t,x,y) \end{aligned} \tag{12}$$

which we call the *transformed* system. We denote the solutions of (12) by $\psi(t,x,y,u^h)$ with $\psi(0,x,y,u^h)=(x,y)^T$.

For $\varepsilon>0$ a *radial retraction* r_ε is a function $r_\varepsilon : X\times Y\to \mathrm{cl}B_\varepsilon(0)$, defined by

$$r_\varepsilon(x,y):=\begin{cases}(x,y) & \text{for } \|(x,y)\|\le\varepsilon\\ \frac{\varepsilon}{\|(x,y)\|}(x,y) & \text{for } \|(x,y)\|>\varepsilon\end{cases}$$

The radial retraction r_ε is uniformly Lipschitz continuous with global Lipschitz constant 1 (see for example Amann [8]). We define

$$\begin{aligned}F_\varepsilon^+(t,x,y) &:= F^+(t,r_\varepsilon(x,y)),\\ F_\varepsilon^-(t,x,y) &:= F^-(t,r_\varepsilon(x,y)),\end{aligned}$$

and get for every $\varepsilon>0$ the *reduced standard system*

$$\begin{aligned}\dot{x} &= Ax+F_\varepsilon^+(t,x,y)\\ \dot{y} &= By+F_\varepsilon^-(t,x,y)\end{aligned} \tag{13}$$

on $X\times Y$ which coincides on $\mathbb{R}\times\mathrm{cl}B_\varepsilon(0)$ with the transformed system (12). We denote the solutions of (13) by

$$\mu_\varepsilon(t,x,y,u^h)=(\mu_{\varepsilon,X}(t,x,y,u^h),\mu_{\varepsilon,Y}(t,x,y,u^h))\in X\times Y$$

with $\mu_\varepsilon(0,x,y,u^h)=(x,y)$. Note that for every $L>0$ we can find an $\varepsilon>0$ such that

$$\begin{aligned}\|F_\varepsilon^+(t,x,y)-F_\varepsilon^+(t,x',y')\| &\le L\,\|(x,y)-(x',y')\|\,,\\ \|F_\varepsilon^-(t,x,y)-F_\varepsilon^-(t,x',y')\| &\le L\,\|(x,y)-(x',y')\|\,,\end{aligned}$$

for all $(x,y),(x',y')\in X\times Y$ and all $t\in\mathbb{R}$.

Convention:

For every $L>0$ we choose $\varepsilon(L)>0$ such that the nonlinearities $F_{\varepsilon(L)}^+$ and $F_{\varepsilon(L)}^-$ have L as common Lipschitz constant and such that the function

$$\varepsilon:\mathbb{R}\to\mathbb{R},L\mapsto\varepsilon(L)$$

is monotonically decreasing. In particular we have $\lim_{L\to 0}\varepsilon(L)=0$.

By choosing ε^* small enough, for every $\varepsilon \in (0, \varepsilon^*]$ we can apply Theorem 6.1.13 in [2] to the reduced system (13) with $\varepsilon := \varepsilon(L)$. We get the existence of stable and unstable fibre bundles.

Theorem 8 *For every $\varepsilon \in (0, \varepsilon^*]$ there are two continuous maps*

$$\begin{aligned} w_\varepsilon^+ &: \mathbb{R} \times X \to Y, \\ w_\varepsilon^- &: \mathbb{R} \times Y \to X. \end{aligned}$$

with $w_\varepsilon^+(t,0) = 0, w_\varepsilon^-(t,0) = 0$ for all $t \in \mathbb{R}$ such that the following invariance equalities holds:

$$\begin{aligned} \mu_{\varepsilon,Y}(t,x,y,u^h) &= w_\varepsilon^+(t, \mu_{\varepsilon,X}(t,x,y,u^h)) \\ \mu_{e,X}(t,x',y',u^h) &= w_\varepsilon^-(t, \mu_{\varepsilon,Y}(t,x',y',u^h)) \end{aligned}$$

for all $t \in \mathbb{R}, (x,y), (x',y) \in X \times Y$ with $y = w_\varepsilon^+(0,x), x' = w_\varepsilon^-(0,y')$. The maps $w_\varepsilon^+, w_\varepsilon^-$ are 2Θ-periodic in the t-component. Furthermore the unstable fibre bundle *$\mathcal{X}_\varepsilon \subset \mathbb{R} \times X \times Y$ and the* stable fibre bundle *$\mathcal{Y}_\varepsilon \subset \mathbb{R} \times X \times Y$ for the reduced system (13) are defined by*

$$\begin{aligned} \mathcal{X}_\varepsilon &:= \{(\tau,x,y) \in \mathbb{R} \times X \times Y : y = w_\varepsilon^+(\tau,x)\}, \\ \mathcal{Y}_\varepsilon &:= \{(\tau,x,y) \in \mathbb{R} \times X \times Y : x = w_\varepsilon^-(\tau,y)\}. \end{aligned}$$

We define $\mathcal{X}_\varepsilon(t) := \{(x,y) : (\tau,x,y) \in \mathcal{X}_\varepsilon\}$ and $\mathcal{Y}_\varepsilon(t) := \{(x,y) : (\tau,x,y) \in \mathcal{Y}_\varepsilon\}$ for $t \in \mathbb{R}$. There are $C_1, C_2, \alpha, \beta > 0$ such that for every $t \in \mathbb{R}, (x,y) \in \mathcal{X}_\varepsilon(0), (x',y') \in \mathcal{Y}_\varepsilon(0)$, we get

$$\begin{aligned} \mu_\varepsilon(t,x,y,u^h) &\in \mathcal{X}_\varepsilon(t), \\ \mu_\varepsilon(t,x',y',u^h) &\in \mathcal{Y}_\varepsilon(t), \end{aligned}$$

and

$$\begin{aligned} \left\|\mu_\varepsilon(t,x,y,u^h)\right\| &\leq C_1 \|x\| e^{\gamma t} \text{ for } t \leq 0, \\ \left\|\mu_\varepsilon(t,x',y',u^h)\right\| &\leq C_2 \|y'\| e^{\gamma t} \text{ for } t \geq 0, \end{aligned} \tag{14}$$

for every $\gamma \in (\lambda_{n+1}^h + \alpha, \lambda_n^h - \beta)$.

PROOF. The existence of the continuous maps w_ε^+ and w_ε^- and the stable and unstable manifolds are given by Theorem 4.16 in [6]. The relations (14) are given by Corollary 6.1.19 in [2]. $\square$

Note that we have $\mathcal{X}_\varepsilon(2k\Theta + t) = \mathcal{X}_\varepsilon(t)$ and $\mathcal{Y}_\varepsilon(2k\Theta + t) = \mathcal{Y}_\varepsilon(t)$ for every $t \in \mathbb{R}, k \in \mathbb{Z}$. For the original system (10) the *local unstable* fibre bundle $\mathcal{X}_\varepsilon^{loc} \subset \mathbb{R} \times \mathbb{R}^d$ and the *local stable* fibre bundle $\mathcal{Y}_\varepsilon^{loc} \subset \mathbb{R} \times \mathbb{R}^d$ are defined by

$$\begin{aligned} \mathcal{X}_\varepsilon^{loc} &:= \{(\tau, p) \in \mathbb{R} \times \mathbb{R}^d : (\tau, \mathcal{F}(\tau)p) \in \mathcal{X}_\varepsilon\}, \\ \mathcal{Y}_\varepsilon^{loc} &:= \{(\tau, p) \in \mathbb{R} \times \mathbb{R}^d : (\tau, \mathcal{F}(\tau)p) \in \mathcal{Y}_\varepsilon\}. \end{aligned}$$

Note, that the local stable and unstable fibre bundles do not have to be invariant. For every subsets $\mathcal{S}$ of $\mathcal{X}_\varepsilon^{loc}$ we will denote in the following $\mathcal{S}(t) := \{p \in \mathbb{R}^d : (t, p) \in \mathcal{S}\}$.

The concept of the *asymptotic phases* goes back to A.M. Lyapunov [9], where he developed this for periodic solutions of analytic differential equations. For invariant manifolds with asymptotic phases and further references see B.Aulbach [10].

Theorem 9 *For every $\varepsilon \in (0, \varepsilon^*]$ there is a continuous mapping $\mathcal{P}_\varepsilon : X \times Y \to \mathcal{X}_\varepsilon(0)$, with the following properties:*

(a) There is a $C > 0$ such that for all $(x, y) \in X \times Y$ the inequality

$$\|\mathcal{P}_\varepsilon(x, y)\| \le C \, \|(x, y)\|, \tag{15}$$

is satisfied. Furthermore for all $(x, y) \in \mathcal{Y}_\varepsilon(0)$ we have $\mathcal{P}(x, y) = 0$.

(b) For every $(x, y) \in \mathcal{X}_\varepsilon(0)$ and every $k \in \mathbb{Z}$ we have

$$\mathcal{P}_\varepsilon(\mu_\varepsilon(2k\Theta, x, y, u^h)) \in \mathcal{X}_\varepsilon(0).$$

PROOF. By Theorem 4.28 in Siegmund [6] we get the existence of the asymptotic phase. $\square$

For the hyperbolic system (10), we define a local version $\mathcal{P}_\varepsilon^{loc}$ of $\mathcal{P}_\varepsilon$ by

$$\begin{aligned} \mathcal{P}_\varepsilon^{loc} : \mathbb{R}^d &\to \quad \mathcal{X}_\varepsilon^{loc}(0) \\ p &\mapsto \mathcal{F}^{-1}(0)\mathcal{P}_\varepsilon(\mathcal{F}(0)p)). \end{aligned} \tag{16}$$

For the construction of the control function u we have to define the set, to which we want to steer with control function u^h (cf. Figure 2). We call this set the *target set* and it will be a subset of $\mathcal{X}_\varepsilon^{loc}$. For the linear system, the target T_ρ is just the *sphere with radius ρ in X*, defined by $T_\rho := \{(x,0) \in X \times Y : \|x\| = \rho\}$ and the *disc D_ρ in X with radius ρ* is $D_\rho := \{(x,0) \in X \times Y : \|x\| \leq \rho\}$. We will extend this definition first to the level of the reduced system (13) and then to the level of the hyperbolic system (10).

Definition 10 *For every $\varepsilon \in (0, \varepsilon^*]$ and for every $\rho > 0$ we define the target $\mathcal{T}_{\varepsilon,\rho}$ and the target disc $\mathcal{D}_{\varepsilon,\rho}$ by*

$$\begin{aligned}\mathcal{T}_{\varepsilon,\rho} &:= \{(t, x, w_\varepsilon^+(t,x)) \in \mathcal{X}_\varepsilon : (x, w_\varepsilon^+(t,x)) \in \mathcal{H}_\varepsilon^{-1}(t, T_\rho)\}\\ \mathcal{D}_{\varepsilon,\rho} &:= \{(t, x, w_\varepsilon^+(t,x)) \in \mathcal{X}_\varepsilon : (x, w_\varepsilon^+(t,x)) \in \mathcal{H}_\varepsilon^{-1}(t, D_\rho)\}\end{aligned}$$

Note, that since $\mathcal{H}_\varepsilon$ is 2Θ-periodic, we have $\mathcal{T}_{\varepsilon,\rho}(t)$ and $\mathcal{D}_{\varepsilon,\rho}(t)$ are 2Θ-periodic in t. $\mathcal{D}_{\varepsilon,\rho}([a,b])$ is pathconnected. If $n = 1$, then $\mathcal{T}_{\varepsilon,\rho}([a,b])$ consists of two continuous curves, and if $d \geq 2$, then $\mathcal{T}_{\varepsilon,\rho}([a,b])$ is pathconnected, too.

Remark 11 *For every $\varepsilon \in (0, \varepsilon^*]$ and every $\sigma > 0$ there exists an $\rho > 0$ such that*

$$\mathcal{D}_{\varepsilon,\rho}(t) \subset \mathcal{X}_\varepsilon(t) \cap B_\sigma(0) \text{ for all } t \in \mathbb{R}.$$

This follows from Corollary 3.2.8 in [2].

Finally we define the target for the hyperbolic system (10).

Definition 12 *For every $\varepsilon \in (0, \varepsilon^*]$ and every $\rho > 0$ we define the* target $\mathcal{T}_{\varepsilon,\rho}^{loc}$ *and the* target disc $\mathcal{D}_{\varepsilon,\rho}^{loc}$

$$\begin{aligned}\mathcal{T}_{\varepsilon,\rho}^{loc} &:= \{(t,p) \in \mathbb{R} \times \mathbb{R}^d : (t, \mathcal{F}(t)p) \in \mathcal{T}_{\varepsilon,\rho}\}\\ \mathcal{D}_{\varepsilon,\rho}^{loc} &:= \{(t,p) \in \mathbb{R} \times \mathbb{R}^d : (t, \mathcal{F}(t)p) \in \mathcal{D}_{\varepsilon,\rho}\}\end{aligned}$$

For $\dim X = 1$, the target $\mathcal{T}_{\varepsilon,\rho}^{loc}$ and the target disc $\mathcal{D}_{\varepsilon,\rho}^{loc}$ are sketched in Figure 3. $\mathcal{T}_{\varepsilon,\rho}(t)$ and $\mathcal{D}_{\varepsilon,\rho}(t)$ are 2Θ-periodic in t.

Remark 13 *There is an $\hat{\varepsilon} \in (0, \varepsilon^*]$ such that for every $\varepsilon \in (0, \hat{\varepsilon}]$ there is a neighborhood $W(\varepsilon) \subset \mathbb{R}^d$ of 0 such that for all $p \in \mathcal{X}_\varepsilon^{loc}(0) \cap W(\varepsilon)$ and all $q \in \mathcal{Y}_\varepsilon^{loc}(0) \cap W(\varepsilon)$ we have*

$$\begin{aligned}\varphi(t,p,u^h) &= \mathcal{F}^{-1}(t)\psi(t, \mathcal{F}(\tau)p, u^h) = \mathcal{F}^{-1}(t)\mu_\varepsilon(t, \mathcal{F}(\tau)p, u^h) \text{ for all } t \leq 0,\\ \varphi(t,q,u^h) &= \mathcal{F}^{-1}(t)\psi(t, \mathcal{F}(\tau)q, u^h) = \mathcal{F}^{-1}(t)\mu_\varepsilon(t, \mathcal{F}(\tau)q, u^h) \text{ for all } t \geq 0.\end{aligned}$$

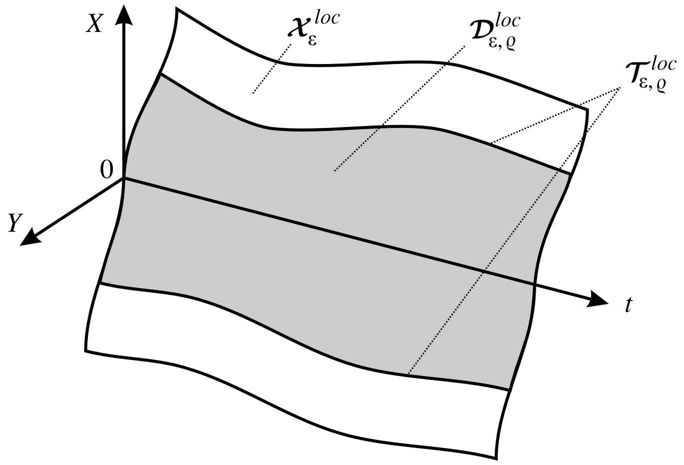


Fig. 3. The target $\mathcal{T}_{\varepsilon,\rho}^{loc}$ and the target disc $\mathcal{D}_{\varepsilon,\rho}^{loc}$.

Then, by Remark 11, for every $\varepsilon \in (0, \hat{\varepsilon}]$ *there is a* $\rho^*(\varepsilon)$ *such that for every* $\rho \in (0, \rho^*(\varepsilon)]$ *and every* $\tau \in \mathbb{R}$ *we have*

$$\mathcal{D}_{\varepsilon,\rho}^{loc}(t) \subset W(\varepsilon).$$

The following result will be used for the construction of the control function in Section 6.

Proposition 14 *Let* $\varepsilon \in (0, \hat{\varepsilon}], \rho \in (0, \rho^*(\varepsilon)], \tau \in \mathbb{R}$ *and let* $c : [0, \infty) \to \mathbb{R}^d$ *be a continuous curve with* $c(t) \in \mathcal{X}_{\varepsilon,\rho}^{loc}(0)$ *for all* $t \geq 0$ *and* $\lim_{t\to\infty} c(t) = 0$. *Then for every* $K_0 \in \mathbb{N}$ *and for every* $S_0 > 0$ *there is a time* $S \geq S_0$ *and a* $k \in \mathbb{N}$ *with* $k > K_0$ *such that*

$$c(S) \in \varphi(-2k\Theta, \mathcal{T}_{\varepsilon,\rho}^{loc}(0), u^h).$$

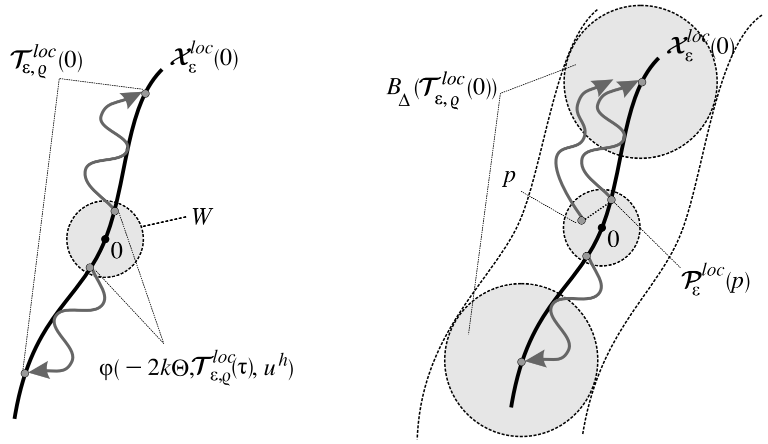
PROOF. For a proof cf. [2], Proposition 3.2.11. □

Next we show, that if we apply the solution map of the hyperbolic system (10) to the target and go backwards in time, then the target shrinks in a uniform way towards the singular point 0. Furthermore, if we start at a point close enough to the shrunk set and follow the trajectory in positive time, then we get into a given neighborhood around the target.

Proposition 15 *Let* $\varepsilon \in (0, \hat{\varepsilon}], \rho \in (0, \rho^*(\varepsilon)]$. *For* $\Delta, T > 0$ *there is a neighborhood* $W := W(\Delta, T, \varepsilon, \rho) \subset \mathbb{R}^d$ *of* 0 *such that*

(a) For every $q \in \mathcal{T}_{\varepsilon,\rho}^{loc}(0)$ *there is a* $k_0 \in \mathbb{N}$ *with* $2k_0\Theta > T$ *such that for every* $k \geq k_0$, $k \in \mathbb{N}$ *we have*

$$\varphi(-2k\Theta, q, u^h) \in W.$$



(a) (b)

Fig. 4. Local behavior of the hyperbolic system.

(b) If $p \in W$ *and* $\mathcal{P}^{loc}_{\varepsilon}(p) = \varphi(-2k\Theta, q, u^h)$*, then*

$$\varphi(t, p, u^h) \in B_\Delta(\mathcal{X}^{loc}_{\varepsilon}(t))$$

for every $t \in [0, 0 + 2k\Theta]$ *and*

$$\varphi(2k\Theta, p, u^h) \in B_\Delta(\mathcal{T}^{loc}_{\varepsilon,\rho}(0)).$$

For an illustration of this result in the case $\dim X = 1$ consider Figure 4.

PROOF. The proof is given in [2], Proposition 3.3.3. □

5 Adjusting the neighborhoods

Before we start with the construction of the control function as explained in Section 3, we have to fix the neighborhoods which result from the linearization of the nonlinear control system. Only in these neighborhoods we are able to characterize the nonlinear system.

First we choose an $\varepsilon \in (0, \hat{\varepsilon}]$, which means that there is a neighborhood $W(\varepsilon) \subset \mathbb{R}^d$ of 0 such that for all $p \in \mathcal{X}^{loc}_{\varepsilon}(0) \cap W(\varepsilon)$ and all $q \in \mathcal{Y}^{loc}_{\varepsilon}(0) \cap W(\varepsilon)$ we have

$$\begin{aligned} \varphi(t, p, u^h) &= \mathcal{F}^{-1}(t)\psi(t, \mathcal{F}(0)p, u^h) = \mathcal{F}^{-1}(t)\mu_\varepsilon(t, \mathcal{F}(0)p, u^h) \text{ for all } t \leq 0, \\ \varphi(t, q, u^h) &= \mathcal{F}^{-1}(t)\psi(t, \mathcal{F}(0)q, u^h) = \mathcal{F}^{-1}(t)\mu_\varepsilon(t, \mathcal{F}(0)q, u^h) \text{ for all } t \geq 0, \end{aligned} \tag{17}$$

according to Remark 13. Note that the system

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i^s(t) f_i(x) \tag{18}$$

(which we call the *stable* system) is locally asymptotically stable (cf. Lemma 6.2.10 in [2]). Thus we can choose a neighborhood $V^s \subset \mathbb{R}^d$ of 0 such that

$$V^s \subset W(\varepsilon)$$

and an open neighborhood $W^s \subset \mathbb{R}^d$ of 0, such that for all $p \in W^s$ we have

$$\varphi(t, p, u^s) \in V^s \text{ for all } t \geq 0 \text{ and } \lim_{t \to \infty} \varphi(t, p, u^s) = 0. \tag{19}$$

By Remark 13 there is a $\hat{\rho} := \hat{\rho}(\varepsilon, W^s) \in (0, \rho^*(\varepsilon))$ such that

$$\mathcal{D}^{loc}_{\varepsilon,\rho}(t) \subset W^s \text{ for all } \rho \in (0, \hat{\rho}) \text{ and all } t \in \mathbb{R}. \tag{20}$$

Thus for all $\rho \in (0, \hat{\rho})$ we obtain

$$\mathcal{T}^{loc}_{\varepsilon,\rho}(t) \subset W^s \subset V^s \subset W(\varepsilon) \text{ for all } t \in \mathbb{R}. \tag{21}$$

Furthermore, we assume that the nonlinear control system is locally accessible on $\mathbb{R}^d \setminus \{0\}$.

6 The Construction of the Control Function

Now we accomplish the construction of the control function. As mentioned in Section 3, the idea is to switch between u^s and u^h (consider Figure 2). We do this in such a way, that the corresponding trajectory gets closer and closer to the target, if we steer the system away form the singular point 0. On the other hand, we steer the trajectory each time closer to the singular point 0, if we apply u^s.

Fix $\varepsilon \in (0, \hat{\varepsilon}], W^s \subset \mathbb{R}^d$ and $\rho \in (0, \hat{\rho})$.

Choose a sequence $(\delta_i)_{i \in \mathbb{N}} \subset \mathbb{R}^+$ with $\lim_{i \to \infty} \delta_i = 0$ and

$$B_{\delta_i}(\mathcal{D}^{loc}_{\varepsilon,\rho}(t)) \subset W^s \text{ for all } i \in \mathbb{N}, t \in \mathbb{R}$$

which is possible because of relation (20).

According to Proposition 15 there exists a sequence $(\sigma_i)_{i\in\mathbb{N}} \subset \mathbb{R}^+$ with $\lim_{i\to\infty}\sigma_i = 0$, such that we have: For every $q \in \mathcal{T}^{loc}_{\varepsilon,\rho}(0)$ there is a $K_0 \in \mathbb{N}$ with $2K_0\Theta > 2^i$ such that for every $k \geq K_0$, $k \in \mathbb{N}$ we have $\varphi(-2k\Theta, q, u^h) \in B_{\sigma_i}(0)$. Furthermore, if $p \in B_{\sigma_i}(0)$ and $\mathcal{P}^{loc}_{\varepsilon}(p) = \varphi(-2k\Theta, q, u^h)$, then

$$\begin{aligned} \varphi(t, p, u^h) &\in B_{\delta_{i+1}}(\mathcal{X}^{loc}_{\varepsilon}(t)) \text{ for every } t \in [0, 0 + 2k\Theta] \text{ and} \\ \varphi(2k\Theta, p, u^h) &\in B_{\delta_{i+1}}(\mathcal{T}^{loc}_{\varepsilon,\rho}(0)). \end{aligned}$$

Now we define the control function recursively, by counting the variable i. We start with $i = 0$

at an arbitrary point $p := p_{i-1} \in \mathcal{T}^{loc}_{\varepsilon,\rho}(0)$.

We define the times ΔT_i, the points p_i and the controls $u_i \in \mathcal{U}$ for $i = 1, 2, \ldots$ recursively.

- By construction, we have $p_{3i-1} \in B_{\delta_i}(\mathcal{T}^{loc}_{\varepsilon,\rho}(\tau))$. Then there is a time $\Delta\tilde{T}_{3i} > 2^i$ with

$$\varphi(t, p_{3i-1}, u^s) \in B_{\sigma_i}(0) \text{ for all } t \geq \Delta\tilde{T}_{3i}.$$

Consider Figure 5 (a).◇

Now there are two cases:

- *First case: For all* $t \geq \Delta\tilde{T}_{3i}$ *we have* $\mathcal{P}^{loc}_{\varepsilon}(\varphi(t, p_{3i-1}, u^s)) \neq 0$.

For this case consider Figure 5 (b). Then we have $\varphi(t, p_{3i-1}, u^s) \notin \mathcal{Y}^{loc}_{\varepsilon}(0)$ for all $t \geq \Delta\tilde{T}_{3i}$, and $\lim_{t\to\infty}\mathcal{P}^{loc}_{\varepsilon}(\varphi(t, p_{3i-1}, u^s)) = 0$. Thus by Proposition 14 it follows, that there is a $k_i \in \mathbb{N}$ with $2k_i\Theta > 2^i$ and $\Delta T_{3i} > \Delta\tilde{T}_{3i}, q_i \in \mathcal{T}^{loc}_{\varepsilon,\rho}(0)$ such that

$$\mathcal{P}^{loc}_{\varepsilon}(\varphi(\Delta T_{3i}, p_{3i-1}, u^s)) = \varphi(-2k_i\Theta, q_i, u^h).$$

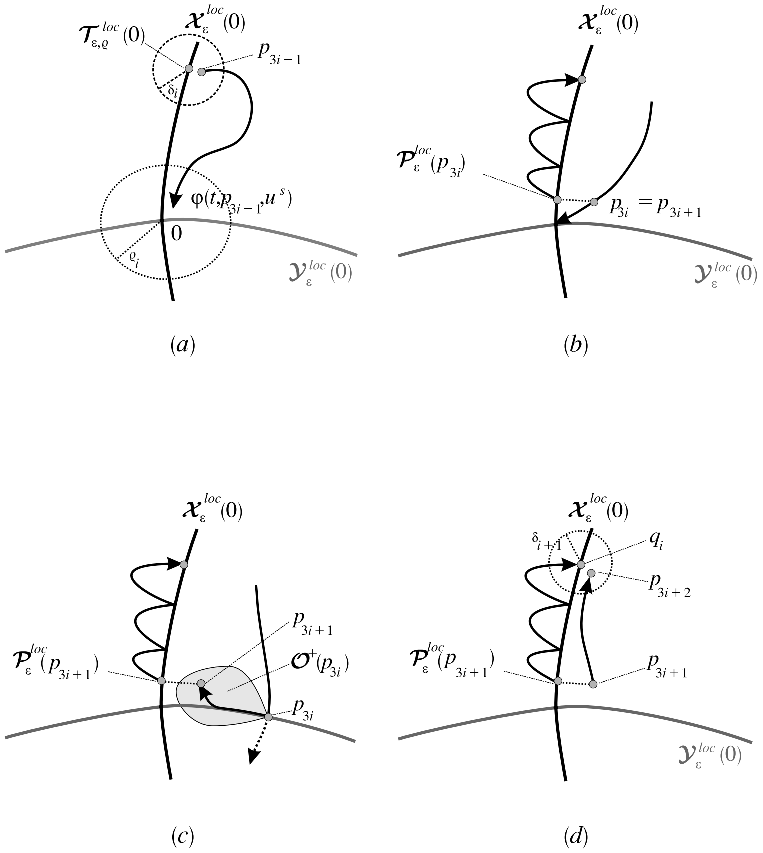


Fig. 5. Illustration of the construction steps.

Define

$$
\begin{aligned}
p_{3i} &:= p_{3i+1} := \varphi(\Delta T_{3i}, p_{3i-1}, u^s),\\
\Delta T_{3i+1} &:= 0,\\
&\text{and } u_i \in \mathcal{U} \text{ arbitrarily.}
\end{aligned}
$$

This completes the first case.$\diamondsuit$

- *Second case: There is a* $\Delta T_{3i} \geq \Delta\tilde{T}_{3i}$ *such that* $P^{loc}_{\varepsilon}(\varphi(\Delta T_{3i}, p_{3i-1}, u^s)) = 0$.

For this case consider Figure 5 (c).Then we define

$$p_{3i} := \varphi(\Delta T_{3i}, p_{3i-1}, u^s)$$

with $p_{3i} \in \mathcal{Y}^{loc}_{\varepsilon}(0)$. Since we assumed local accessibility on $\mathbb{R}^d \setminus \{0\}$ there is a control function $u_i \in \mathcal{U}$ and a time $0 < \Delta\tilde{T}_{3i+1} < 2^{-i}$, such that

$$\begin{aligned} \varphi(t, p_{3i}, u_i) &\in B_{\sigma_i}(0) \text{ for all } t \in [0, \Delta\tilde{T}_{3i+1}] \text{ and} \\ \varphi(\Delta\tilde{T}_{3i+1}, p_{3i}, u_i) &\notin \mathcal{Y}_\varepsilon^{loc}(0). \end{aligned}$$

Then there is an interval $[a_i, b_i] \subset [0, \Delta\tilde{T}_{3i+1}]$ with

$$\begin{aligned} \varphi(a_i, p_{3i}, u_i) &\in \mathcal{Y}_\varepsilon^{loc}(0) \text{ and} \\ \varphi(t, p_{3i}, u_i) &\notin \mathcal{Y}_\varepsilon^{loc}(0) \text{ for all } t \in (a_i, b_i]. \end{aligned}$$

Because $\varphi(a_i, p_{3i}, u_i) \in \mathcal{Y}_\varepsilon^{loc}(0)$ we have

$$\mathcal{P}_\varepsilon^{loc}(\varphi(a_i, p_{3i}, u_i)) = 0,$$

and because $\varphi(t, p_{3i}, u_i) \notin \mathcal{Y}_\varepsilon^{loc}(0)$ we get for all $t \in (a_i, b_i]$

$$\mathcal{P}_\varepsilon^{loc}(\varphi(t, p_{3i}, u_i)) \in \mathcal{X}_\varepsilon^{loc}(0) \setminus \{0\}.$$

Thus, according to Proposition 14, there is a $k_i \in \mathbb{N}$ with $2k_i\Theta > 2^i$ and $q_i \in \mathcal{T}_{\varepsilon,\rho}^{loc}(0), \Delta T_{3i+1} \in (a_i, b_i]$ such that

$$\mathcal{P}_\varepsilon^{loc}(\varphi(\Delta T_{3i+1}, 0, p_{3i}, u_i)) = \varphi(-2k_i\Theta, q_i, u^h).$$

Define

$$p_{3i+1} := \varphi(\Delta T_{3i+1}, p_{3i}, u_i).$$

This completes the second case.$\diamondsuit$

• Now in both cases, we stopped with a point $p_{3i+1} \in B_{\rho_i}(0) \cap \mathcal{X}_\varepsilon^{loc}(0)$ and a $q_i \in \mathcal{T}_{\varepsilon,\rho}^{loc}(0)$ such that

$$\mathcal{P}_\varepsilon^{loc}(p_{3i+1}) = \varphi(-2k_i\Theta, q_i, u^h).$$

Consider Figure 5 (d). Define

$$\begin{aligned} \Delta T_{3i+2} &:= 2k_i\Theta \text{ and} \\ p_{3i+2} &:= \varphi(\Delta T_{3i+2}, p_{3i+1}, u^h) = \varphi(\Delta T_{3i+2}, p_{3i+1}, u^h). \end{aligned}$$

By construction, $p_{3i+2} \in B_{\delta_{i+1}}(\mathcal{T}^{loc}_{\varepsilon,\rho}(\Delta T_{3i+2})) = B_{\delta_{i+1}}(\mathcal{T}^{loc}_{\varepsilon,\rho}(0))$. This is the end of the construction.$\diamondsuit$

We define

$$T_i := \sum_{k=0}^{i} \Delta T_k$$

and the function $u : \mathbb{R} \to U$ by

$$u(t) := \begin{cases} 0 & \text{for } t < 0, \\ u^s(t - T_{3i-1}) & \text{for } t \in [T_{3i-1}, T_{3i}), \\ u_i(t - T_{3i}) & \text{for } t \in [T_{3i}, T_{3i+1}), \\ u^h(t - T_{3i+1}) & \text{for } t \in [T_{3i+1}, T_{3i+2}), \end{cases}$$

for $i = 0, 1, 2, \ldots$ and $T_{-1} := 0$.

Remark 16 *The function u depends on the chosen constants ε, ρ and the starting point $p \in \mathcal{T}^{loc}_{\varepsilon,\rho}(0)$. Therefore all the times ΔT_i and the control functions u_i depend on p, ε and ρ. For indicating this, we will denote u by $u_{\varepsilon,\rho,p}$. But $u_{\varepsilon,\rho,p}$ is* not *uniquely determined, because we may choose the ΔT_{3i} as large as we want.*

Remark 17 *By local uniqueness of the unstable and stable fibre bundles (cf. Proposition 6.2.11 in [2]) it follows, that for a given $\varepsilon \in (0, \hat{\varepsilon}]$ for every $\varepsilon' \in [\varepsilon, \hat{\varepsilon}]$ we can construct $u_{\varepsilon,\rho,p}$ and $u_{\varepsilon',\rho,p}$ such that $u_{\varepsilon,\rho,p} = u_{\varepsilon',\rho,p}$.*

7 The ω-Limit Set

We want to apply Proposition 6 to get the existence of a control set with nonvoid interior. The proposition demands the existence of a compact subset of $\mathcal{U} \times \mathbb{R}^d$ which has nonvoid intersection with the ω-limit set $\omega(u, x)$, on which two inner pair conditions are satisfied.

If we take a pair $(u^*, p^*) \in \omega(u, p)$, then for every $-\infty < T_1 \leq T_2 < \infty$ the set

$$\{\Phi_t(u^*, p^*) : t \in [T_1, T_2]\}$$

is a compact subset of $\omega(u, p)$. This will be used in the next section to proof the Existence Theorem 20 for control sets near the singular point $x^* = 0$.

Here we show that the ω-limit set has nonvoid intersection with the target $\mathcal{T}_{\varepsilon,\rho}^{loc}(\tau)$.

Theorem 18 *Let* $p \in \mathcal{T}_{\varepsilon,\rho}^{loc}(0)$ *and let* $u := u_{\varepsilon,\rho,p} \in \mathcal{U}$ *be constructed as in Section 6. Then there is a* $p^* \in \mathcal{T}_{\varepsilon,\rho}^{loc}(0) \subset \mathcal{X}_{\varepsilon}^{loc}(0)$ *such that*

$$\{\Phi_\tau(u^*, p^*) : \tau \in \mathbb{R}\} \subset \omega(u, p)$$

where $u^* : \mathbb{R} \to U$ *is defined by*

$$u^*(t) := \begin{cases} u^s(t) \text{ for } t \geq 0, \\ u^h(t) \text{ for } t < 0. \end{cases}$$

PROOF. First we show, that

$$\lim_{k\to\infty} \theta_{T_{3k-1}} u = u^*.$$

For that purpose, remind that $\mathcal{U}$ is supplied with the weak*-topology of $L_\infty(\mathbb{R}, \mathbb{R}^m)$. Let $W \subset \mathcal{U}$ be a neighborhood of u^*. Then there exists a $\sigma > 0$ and $g_1, \ldots, g_n \in L^1(\mathbb{R}, \mathbb{R}^m)$ such that

$$\left\{ v \in L^\infty(\mathbb{R}, \mathbb{R}^m) : \begin{array}{c} |\int_{\mathbb{R}} \langle u^*(t) - v(t), g_j(t)\rangle \, dt| < \sigma \\ \text{for } j = 1, \ldots, n, \text{ and } v(t) \in U, \forall t \in \mathbb{R} \end{array} \right\} \subset W, \qquad (22)$$

because the sets of this form are a subbases of the weak*-topology (cf. for example Dunford and Schwartz [11]). We show, that for all $g \in L^1(\mathbb{R}, \mathbb{R}^m)$ and all $\sigma > 0$ there is a $N \in \mathbb{N}$ with

$$\left| \int_{\mathbb{R}} \langle u^*(t) - \theta_{T_{3k}} u(t), g(t)\rangle \, dt \right| < \sigma \text{ for all } k > N.$$

Then it follows that $\theta_{T_{3k}} u$ is an element of the set on the left hand side of (22) and hence $\theta_{T_{3k}} u \in W$ for all $k \in \mathbb{N}$ big enough.

So let $g \in L^1(\mathbb{R}, \mathbb{R}^m)$ and $\sigma > 0$. Then there exists a time $T > 0$ with

$$\int_{\mathbb{R}\setminus[-T,T]} |g(t)| \, dt < \frac{\sigma}{2\mathrm{diam} U}.$$

Furthermore, because $\Delta T_{3i} > 2^k$ and $\Delta T_{3i-1} > 2^k$ for all $k \in \mathbb{N}$ by construction of the control function $u \in \mathcal{U}$, there exists a $N \in \mathbb{N}$ with

$$\begin{aligned} \Delta T_{3k-1} &> T \text{ and} \\ \Delta T_{3k} &> T \text{ for all } k > N. \end{aligned}$$

This guarantees, that

$$\begin{aligned} &u(t+T_{3k-1}) = u^s(t) \text{ for all } t \in [0,T] \text{ and} \\ &u(t+T_{3k-1}) = u^h(t) \text{ for all } t \in [-T,0]. \end{aligned}$$

Then we get

$$\begin{aligned} &|\textstyle\int_{\mathbb{R}} \langle u(t+T_{3k-1}) - u^*(t), g(t)\rangle\, dt| \\ &\leq \left|\textstyle\int_{\mathbb{R}^-\backslash[-T,0]} \langle u(t+T_{3k-1}) - u^*(t), g(t)\rangle\, dt\right| \\ &+ \left|\textstyle\int_{\mathbb{R}^+\backslash[0,T]} \langle u(t+T_{3k-1}) - u^*(t), g(t)\rangle\, dt\right| \\ &+ \left|\textstyle\int_{[-T,0]} \langle u(t+T_{3k-1}) - u^*(t), g(t)\rangle\, dt\right| \\ &+ \left|\textstyle\int_{[0,T]} \langle u(t+T_{3k-1}) - u^*(t), g(t)\rangle\, dt\right| \\ &\leq \tfrac{\sigma}{2} + \tfrac{\sigma}{2} + \left|\textstyle\int_{[-T,0]} \left\langle u^h(t) - u^*(t), g(t)\right\rangle dt\right| \\ &+ \left|\textstyle\int_{[0,T]} \langle u^s(t) - u^*(t), g(t)\rangle\, dt\right| \\ &= \sigma. \end{aligned}$$

By construction we have for all $k \in \mathbb{N}$

$$\varphi(T_{3k-1}, p, u) \in B_{\delta_{k+1}}(\mathcal{T}^{loc}_{\varepsilon,\rho}(0)).$$

Since $\mathcal{T}^{loc}_{\varepsilon,\rho}(0)$ is compact, there exists a subsequence $(T_{3k_l})_{l\in\mathbb{N}}$ with

$$p^* := \lim_{l\to\infty} \varphi(T_{3k_l-1}, p, u) \in \mathcal{T}^{loc}_{\varepsilon,\rho}(0).$$

Thus it follows, that $(u^*, p^*) \in \omega(u,p)$. Because $\omega(u,p)$ is invariant the assertion follows. □

Remark 19 *The point p^* is* not *uniquely defined. As one sees in the proof, p^* is only given by some convergent subsequence of $(\varphi(T_{3k-1}, p, u))_{k\in\mathbb{N}}$.*

8 The Existence Theorem

After the construction of the control function $u \in \mathcal{U}$ in Section 6 and the characterization of the ω-limit set in Section 7, we finally use this to prove the following existence theorem.

Theorem 20 *Consider the nonlinear control system*

$$\begin{aligned} &\dot{x} = f_0(x) + \textstyle\sum_{i=1}^{m} u_i(t)\, f_i(x) \\ &u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\} \end{aligned} \tag{23}$$

where U is a compact and convex subset of $\mathbb{R}^m$ and $f_0, \ldots, f_m$ are C^2 vector fields on $\mathbb{R}^d$. We assume, that following properties are satisfied.

1. *The nonlinear control system (23) has one singular point $x^* = 0 \in \mathbb{R}^d$, and $\mathbb{R}^d \setminus \{0\}$ and $\{0\}$ are maximal integral manifolds of (23).*
2. *There are periodic control functions u^h and $u^s \in \mathcal{U}$ such that the associated Lyapunov exponents of the linearized systems have the following properties*

   $$\begin{gathered} 0 > \lambda_1^s \geq \ldots \geq \lambda_d^s \quad \text{and} \\ \lambda_1^h \geq \ldots \geq \lambda_k^h > 0 > \lambda_{k+1}^h \geq \ldots \geq \lambda_d^h \text{ for } 1 \leq k < d. \end{gathered}$$

   Then define $\hat{\varepsilon}$ as in Remark 13, choose $\varepsilon \in (0, \hat{\varepsilon}]$ and denote by $\mathcal{X}_\varepsilon^{loc}, \mathcal{Y}_\varepsilon^{loc}$ the corresponding local unstable and stable fibre bundle of the differential equation (23) corresponding to u^h.
3. *There is a neighborhood $V \subset \mathbb{R}^d$ of x^* such that for every $t \in \mathbb{R}$ and every $x \in \mathcal{X}_\varepsilon^{loc}(0) \cap V \setminus \{x^*\}$ the pair $(u^h(t + \cdot), x)$ is a strong inner pair.*

Then there exists a control set $D \subset \mathbb{R}^d$ with nonvoid interior and a $p \in \mathcal{X}_\varepsilon^{loc}(0) \cap V \setminus \{x^*\}$ with

$$\{\varphi(t, p, u^h) : t < 0\} \subset intD.$$

In particular we have $x^* \in clD$. If in addition all the pairs $\{(u^s(t+\cdot), \varphi(t, p, u^s)), t \geq 0\}$ are strong inner pairs, then we also have

$$\{\varphi(t, p, u^s) : t \geq 0\} \subset intD.$$

PROOF. Choose $\varepsilon \in (0, \hat{\varepsilon}]$ and $V^s \subset W(\varepsilon)$ as in Section 5 such that $V^s \subset V$. Next choose the open neighborhood $W^s \subset V^s$ of 0 with (19) and define

$\hat{\rho} = \hat{\rho}(\varepsilon, W^s)$ as in Section 5. Then for $\rho \in (0, \hat{\rho}]$ we can construct the control function u as in Section 6 for a given $p \in \mathcal{T}^{loc}_{\varepsilon,\rho}(0)$. By applying Theorem 18 we find a point $p^* \in \mathcal{T}^{loc}_{\varepsilon,\rho}(0)$ such that

$$\{\Phi_t(u^*, p^*) : t \in \mathbb{R}\} \subset \omega(u, p)$$

where $u^* : \mathbb{R} \to U$ is defined by

$$u^*(t) := \begin{cases} u^s(t) \text{ for } t \geq 0, \\ u^h(t) \text{ for } t < 0. \end{cases}$$

Note that for every $-\infty < T_1 \leq T_2 < \infty$ the sets

$$\{\Phi_t(u^*, p^*) : t \in [T_1, T_2]\} \subset \mathcal{U} \times \mathbb{R}^d$$

are compact.

Choose $-\infty < T_0 < 0$ arbitrarily and a $\sigma > 0$ such that $T_0 + \sigma < 0$. Both $\Phi_{T_0}(u^*, p^*)$ and $\Phi_{T_0+\sigma}(u^*, p^*)$ are compact subsets of $\mathcal{U} \times \mathbb{R}^d$ and are strong inner pairs by assumption, because $\varphi(t, p^*, u^h) \in V^s \subset W(\varepsilon) \cap V$ for all $t \leq 0$. Thus by applying Proposition 6 we find a control set $D \subset \mathbb{R}^d$ with nonvoid interior and

$$\varphi(T_0, p^*, u^h) \subset \text{int} D.$$

We show, that $\varphi(t, p^*, u^h) \in \text{int} D$ for all $t < 0$. Choose $-\infty < T_1 < T_2 < 0$ and $\sigma > 0$ with $T_0 \in [T_1, T_2]$. Then by applying Propositions 6 again, we find a control set $\tilde{D} \subset \mathbb{R}^d$ with $\{\varphi(t, p^*, u^h) : t \in [T_1, T_2]\} \subset \text{int}\tilde{D}$. On the other hand, since $\varphi(T_0, p^*, u^h) \subset \text{int} D$ we have $D \cap \tilde{D} \neq \emptyset$. By the maximality property of the control sets (cf. Definition 2) we get $D = \tilde{D}$.

Next we suppose, that all the pairs $\{(u^s(t+\cdot), \varphi(t, 0, p_\tau, u^s)), t \geq 0\}$ are strong inner pairs. By choosing now $T_2 > 0$, we again get by Proposition 6 that there is a control set $\tilde{D} \subset \mathbb{R}^d$ with $\{\varphi(t, p^*, u^*) : t \in [T_0, T_1]\} \subset \text{int}\tilde{D}$. By maximality of control sets we finally obtain $D = \tilde{D}$. □

Remark 21 *If we suppose in Theorem 20 instead of (3), that there is a neighborhood* $W \subset \mathbb{R}^d$ *of* x^* *such that for every* $t \in \mathbb{R}$ *and every* $x \in W \setminus \{x^*\}$ *the pair* $(u^s(t + \cdot), x)$ *is a strong inner pair, we also get that there exists a control set* $D \subset \mathbb{R}^d$ *with nonvoid interior and a* $p \in \mathcal{X}^{loc}_{\varepsilon}(0) \cap V \setminus \{x^*\}$ *with* $\{\varphi(t, p, u^s) : t \geq 0\} \subset \textit{int} D$.

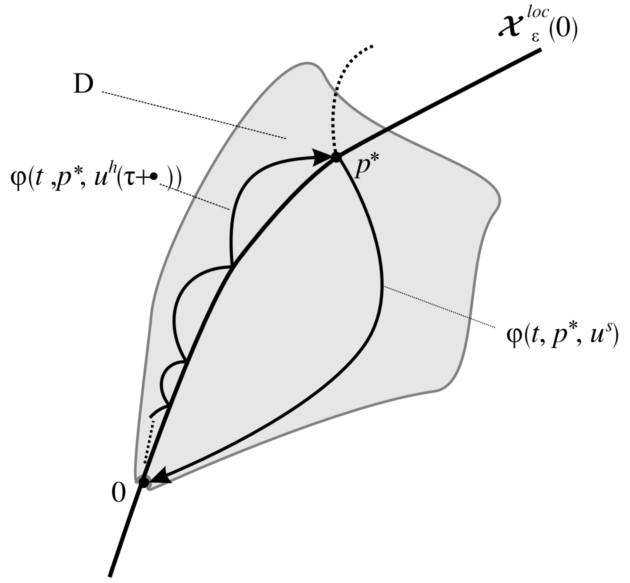


Fig. 6. Illustration of the Existence Theorem.

Remark 22 *Note, that the control set D is not unique. The point $p^* \in \mathcal{T}^{loc}_{\varepsilon,\rho}(0)$ in the proof, which we get by the construction in Theorem 18, is not specified any further (cf. Remark 19). In the case where $\dim \mathcal{X}^{loc}_{\varepsilon,\rho} = 1$, the target $\mathcal{T}^{loc}_{\varepsilon,\rho}(0)$ consist of only two points, and thus the control sets D can be described in more detail (cf. Chapter 4 in [2]).*

Figure 6 illustrates the result of the Theorem 20. If we assume here (for better illustration), that $\dim \mathcal{X}^{loc}_{\varepsilon} = 1$, then for a given $\tau \in \mathbb{R}$, the fibre $\mathcal{X}^{loc}_{\varepsilon}(0)$ is just a continuous curve in $\mathbb{R}^d$ through 0. We assume here, that the additional condition in Theorem 20 (all the pairs $\{(u^s(t+\cdot), \varphi(t,p,u^s)), t \geq 0\}$ are strong inner pairs) is fulfilled. Now the result of the theorem is, that we can find a point on the local unstable fibre bundle, such that if we apply the control function u^* (defined as in Theorem 18), then the corresponding trajectory lies in the interior of a control set D.

Note that we stated the theorem without using the target $\mathcal{T}^{loc}_{\varepsilon,\rho}$. But it is clear, that for every $\rho > 0$ small enough, we get a corresponding control set D_ρ with $\mathcal{T}^{loc}_{\varepsilon,\rho}(0\ ) \cap D_\rho \neq 0$. The trajectory $\varphi(t,p^*,u^*)$ seems to jump out of the local unstable fibre bundle $\mathcal{X}^{loc}_{\varepsilon}(0)$ for $t < 0$. This is due to the fact, that we have drawn here only the fibre $\mathcal{X}^{loc}_{\varepsilon}(0)$. For all $t < 0$ we have

$$\begin{aligned} \varphi(t,p^*,u^*) &= \varphi(t,p^*,u^h) \\ &= \varphi(t,p^*,u^h) \in \mathcal{X}^{loc}_{\varepsilon}(t). \end{aligned}$$

Because $\mathcal{X}^{loc}_{\varepsilon}(t)$ is 2Θ-periodic, we get for all $k \in \mathbb{N}$

$$\varphi(-2k\Theta, p^*, u^h) \in \mathcal{X}^{loc}_{\varepsilon}(-2k\Theta) = \mathcal{X}^{loc}_{\varepsilon}(0).$$

This explains the jumps. $\varphi(t, p^*, u^h)$ is an element of $\mathcal{X}_\varepsilon^{loc}(t)$ but we have drawn here only $\mathcal{X}_\varepsilon^{loc}(0)$. Thus for all $k \in \mathbb{N}$ the trajectory $\varphi(-2k\Theta, p^*, u^h)$ hits $\mathcal{X}_\varepsilon^{loc}(0)$ and for all other times it does not have to lie on $\mathcal{X}_\varepsilon^{loc}(0)$.

Furthermore note, that from Theorem 20 it follows that for every $\tau \in \mathbb{R}$ we can find a control set D_τ and a $p_\tau \in \mathcal{X}_\varepsilon^{loc}(\tau) \cap V \setminus \{x^*\}$ with

$$\{\varphi(t, p_\tau, u^h(\tau + \cdot) : t < 0\} \subset \mathrm{int} D_\tau,$$
$$\{\varphi(t, p_\tau, u^s) : t \geq 0\} \subset \mathrm{int} D_\tau,$$

cf. Theorem 3.7.1 in [2]

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