Bifurcation of Control Sets at Singular Points

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Abstract: We look at a control affine system in $\mathbb{R}^d$ with a singular point $x^* \in \mathbb{R}^d$. Motivated by the example of the perturbed Duffing-van der Pol equation we show, that under a condition on the Lyapunov exponents, there exists a control set $D \subset \mathbb{R}^d$ with nonvoid interior such that $x^* \in closure(D)$.

We look at control affine systems of the form

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i(t) f_i(x) \tag{1}$$
$$u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \,\forall t \in \mathbb{R}, \text{locally integrable}\},$$

where U is a compact and convex subset of $\mathbb{R}^m$ containing 0 and $f_0, \dots, f_m$ are C^2 vector fields on $\mathbb{R}^d$. We denote the solutions of (1) by $\varphi(t, x, u)$ with $\varphi(0, x, u) = x$. We assume that the system has a *singular point* $x^* \in \mathbb{R}^d$, i.e. $f_i(x^*) = 0$ for all $i = 0, \dots, m$. We also assume, that the system is locally accessible for every $\mathbb{R}^d \setminus \{x^*\}$.

Definition 1 *A* control set *D is a subset of $\mathbb{R}^d$ with*

(i) $D \subset closure\mathcal{O}^+(x)$ for all $x \in D$ with $\mathcal{O}^+(x) := \{\varphi(t, x, u) : t \geq 0, u \in \mathcal{U}\}$.

(ii) For all $x \in D$ there is an $u \in \mathcal{U}$ with $\varphi(t, x, u) \in D$ for all $t \geq 0$.

(iii) D is maximal with respect to (i) and (ii).

We refer to [1], Chapter 3 for a full description of the control set concept. The control system (1) has at least one control set, namely the singular point x^*.

Example 2 *Consider the perturbed Duffing-van der Pol equation*

$$\left\{ \begin{array}{l} \dot{x} = y \\ \dot{y} = (-\frac{1}{4} + u(t))x - 2y - x^3 - x^2 y \end{array} \right. \tag{2}$$
$$u \in \mathcal{U}^\rho := \{u : \mathbb{R} \to U^\rho : \textit{locally integrable}\}$$

where $U^\rho := [-\rho, \rho], \rho \in [0, \frac{3}{4})$ and $x^* = 0$ is a singular point. For $\rho \in [0, \frac{1}{4})$ the system (2) has only one control set, namely the singular point $x^* = 0$. By numerical computation we observe, that for $\rho \in (\frac{1}{4}, \frac{3}{4})$ it has in addition two more control sets $D_1^\rho, D_2^\rho \subset \mathbb{R}^2$ with nonvoid interior and $0 \in closure(D_i^\rho), i = 1, 2$.

Now the question arises under which conditions we can assure, that the nonlinear control system (1) has a control set $D \subset \mathbb{R}^d$ with nonvoid interior such that $x^* \in closure(D)$.

First of all, we have to give an existence criterion for control sets. For doing this, we need the notion of the control flow on $\mathcal{U} \times \mathbb{R}^d$.

Proposition 3 *The map*

$$\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d \to \mathcal{U} \times \mathbb{R}^d, \Phi(t, u, x) = (u(t + \cdot), \varphi(t, x, u))$$

defines a continuous dynamical system on $\mathcal{U} \times \mathbb{R}^d$ if we supply $\mathcal{U}$ with the weak-topology of $L^\infty(\mathbb{R}, \mathbb{R}^d)$. It is called the* control flow.

Proof. See Lemma 4.3.2 in [1]. ■

Definition 4 *Let $(u,x) \in \mathcal{U} \times M$ such that $\{\varphi(t,x,u) : t \geq 0\}$ is bounded. The* ω-limit set $\omega(u,x)$ *of (u,x) is defined as*

$$\omega(u,x) := \left\{ (v,y) \in \mathcal{U} \times \mathbb{R}^d : \begin{array}{c} \text{there exists a sequence } \{t_k\}_{k\in\mathbb{N}} \subset \mathbb{R} \\ \text{with } t_k \to \infty \text{ for } k \to \infty \text{ such that} \\ (u(t_k + \cdot), \varphi(t_k, x, u)) \to (v,y) \text{ as } k \to \infty. \end{array} \right\}$$

The *projection* $\pi : \mathcal{U} \times \mathbb{R}^d \to \mathbb{R}^d$ is defined as

$$\pi(u,x) := x$$

Proposition 5 (Existence Criterion) *Let $\mathbb{R}^d \setminus \{x^*\}$ and x^* be maximal integral manifolds. Let $x \in \mathbb{R}^d \setminus \{x^*\}$ and $u \in \mathcal{U}$, such that $\{\varphi(t,x,u(\cdot)) : t \geq 0\}$ is bounded. Suppose further that there is a compactum $\mathcal{K} \subset \mathcal{U} \times \mathbb{R}^d$ such that following holds:*

(i) $\mathcal{K} \subset \mathcal{U} \times \mathbb{R}^d \setminus \{x^*\}$ and $\mathcal{K} \cap \omega(u,x) \neq \emptyset$.

(ii) For all $(v,y) \in \mathcal{K}$ there is a $t > 0$ with $\varphi(t,y,v) \in int\mathcal{O}^+(y)$ (inner pair condition).

(iii) There is an $s > 0$ such that for all $(v,y) \in \Phi_{-s}\mathcal{K}$ and all $t > 0$ it holds that $\varphi(t,y,v) \in int\mathcal{O}^+(y)$ (strong inner pair condition).

Then there exists a control set $D \subset \mathbb{R}^d$ with

$$\pi(\mathcal{K} \cap \omega(u,x)) \subset intD.$$

Proof. Is given by a slightly modification of Corollary 4.5.16 in [1], where only a constant $u \in U$ is allowed. ■

Now our aim is to construct such a trajectory as in the preceding proposition. By linearization of the nonlinear control system (1) at the singular point x^* we get the *bilinear control system*

$$\dot{x} = A_0 x + \sum_{i=1}^{m} u_i(t) A_i(x), \ u \in \mathcal{U}, \text{ where } A_i := \left. \frac{\partial f_i}{\partial x} \right|_{x=x^*}. \tag{3}$$

Denote the solutions of (3) by $\psi(t,x,u)$ with $\psi(0,x,u) = x$.

For every $(u,x) \in \mathcal{U} \times \mathbb{R}^d, x \neq 0$ define the *Lyapunov exponent*

$$\lambda(u,x) := \limsup_{t\to\infty} \frac{1}{t} \ln \|\psi(t,x,u)\|.$$

For given $u \in \mathcal{U}$ the set $\{\lambda(u,x) : x \in \mathbb{R}^d, x \neq 0\}$ consists of d elements $\lambda_1(u), \ldots, \lambda_d(u) \in \mathbb{R}$. We assume, that there are two periodic piecewise constant controls $u^h, u^s \in \mathcal{U}$ such that the corresponding Lyapunov exponents have the properties

$$\begin{aligned} 0 &> \lambda_1(u^s) \geq \ldots \geq \lambda_d(u^s) \text{ and} \\ \lambda_1(u^h) \geq \ldots \geq \lambda_k(u^h) > 0 &> \lambda_{k+1}(u^h) \geq \ldots \geq \lambda_d(u^h), \text{ with } 1 \leq k < d. \end{aligned}$$

This situation occurs for example, if $\rho \in (\frac{1}{2}, \frac{5}{4})$ for the perturbed Duffing- van der Pol equation (2) with $k = 1$.

We use the technique of stable and unstable fibre bundles for *nonautonomous* differential equations, developed by [2] (see also [3]) . If we apply the control function u^h in (1), there exists a *local unstable fibre bundle* for x^*

$$R^{loc}(u^h) \subset \mathbb{R} \times \mathbb{R}^d \text{ with } (t,x^*) \in R^{loc}(u^h) \text{ for all } t \in \mathbb{R}.$$

We denote the fibre of $R^{loc}(u^h)$ at time $t \in \mathbb{R}$ by $R^{loc}(t, u^h) = \{x \in \mathbb{R}^d : (t, x) \in R^{loc}(u^h)\}$. $R^{loc}(u^h)$ can be parametrized locally around x^* by a continuous function $r^{loc} : \mathbb{R} \times W^r \to \mathbb{R}^d$ for an open subset $W^r \subset \mathbb{R}^k$ such that we have

$$R^{loc}(u^h) \cap \mathbb{R} \times V = \{(t, r^{loc}(t, x)) : x \in W^r\}.$$

for an open neighborhood $V \subset \mathbb{R}^d$ of x^*. Vice versa, there is a local stable fibre bundle $S^{loc}(u^h) \subset \mathbb{R} \times \mathbb{R}^d$, which can be parametrized locally around x^* by a continuous function $s^{loc} : \mathbb{R} \times W^s \to \mathbb{R}^d$ for an open subset $W^s \subset \mathbb{R}^{d-k}$.

Before stating the main theorem, we need the notion of *strong inner pairs.* A pair $(u, x) \in \mathcal{U} \times \mathbb{R}^d$ is called strong inner pair, if $\varphi(t, x, u) \in int\mathcal{O}^+(x)$ for all $t > 0$.

Theorem 6 *Assume, that there is a neighborhood $V \subset \mathbb{R}^d$ of x^* such for every $t \in \mathbb{R}$ and every $x \in R^{loc}(t, u^h) \cap V \setminus \{x^*\}$ the pairs $(u^h(t+\cdot), x)$ are strong inner pairs. Then there exist a control set $D \subset \mathbb{R}^d$ with nonvoid interior and a point $p \in R^{loc}(0, u^h)$ such that $\{\varphi(t, p, u^h(t+\cdot)) : t \leq 0\} \subset D$. In particular we have $x^* \in closure(D)$.*

Sketch of the Proof: For showing the existence of a control set, according to Proposition 5 we have to construct a control function $u \in \mathcal{U}$ and find a $p \in \mathbb{R}^d \setminus \{x^*\}$ such that the set $\{\varphi(t, p, u) : t \geq 0\}$ is bounded, but $\lim_{t\to\infty} \varphi(t, p, u) \neq x^*$. We do this, by switching between the two control functions u^s and u^h and starting with a point $p \in \mathbb{R}^d \setminus \{x^*\}$ in a neighborhood of x^* which is small enough.

If we apply the control function u^s to the system (1), the singular point x^* is locally asymptotically stable. Thus by starting at the point p we can get arbitrarily close to x^* by applying u^s.

Now every point in a small neighborhood around x^* which does lie not in a fibre of $S^{loc}(u^h)$ can be steered away from x^* by applying the control function u^h. If a point lies on the stable bundle, then by local accessibility of the nonlinear control system (1) we can apply another control function to get out of the stable fibre bundle. Then by applying u^h we can steer away from x^*. We do this in both cases, until we reach a certain distance of x^*. Now we apply again u^s to get near x^* and then u^h to get away and so on. Thus by switching between u^s and u^h we get a control function $u \in \mathcal{U}$ with the desired properties.■

A special case occurs, if the unstable manifold $R^{loc}(u^h)$ is onedimensional which means that $k = 1$. This is for example the situation for the perturbed Duffing-van der Pol equation (2). Then $R^{loc}(t, u^h)$ can be parametrized for every $t \in \mathbb{R}$ by a continuous curve through x^*. The singular point x^* divides the curve into two pieces which we denote by $\overline{R}^{loc}(t, u^h)$ and $\underline{R}^{loc}(t, u^h)$. In the example 2, we have seen, that there is a control set around each branch $\overline{R}^{loc}(t, u^h)$ and $\underline{R}^{loc}(t, u^h)$ emanating from x^*. By a construction of the control function u slightly different from the one in the proof of Theorem 6 we get

Theorem 7 *Assume, that $k = 1$ and that there is a neighborhood $V \subset \mathbb{R}^d$ of x^* such for every $t \in \mathbb{R}$ and every $x \in R^{loc}(t, u^h) \cap V \setminus \{x^*\}, y \in S^{loc}(t, u^h) \cap V \setminus \{x^*\}, z \in V \setminus \{x^*\}$ the pairs $(u^h(t+\cdot), x), (u^h(t+\cdot), y)$ and $(u^s(t+\cdot), z)$ are strong inner pairs. Then there exist two control sets $\overline{D}, \underline{D} \subset \mathbb{R}^d$ with nonvoid interior and points $\overline{p} \in \overline{R}^{loc}(0, u^h), \underline{p} \in \underline{R}^{loc}(0, u^h)$ such that*

$$\{\varphi(t, \overline{p}, u^h(t+\cdot)) : t \leq 0\} \subset \overline{D}$$
$$\{\varphi(t, \underline{p}, u^h(t+\cdot)) : t \leq 0\} \subset \underline{D}.$$

In particular we have $x^* \in closure(\overline{D}) \cap closure(\underline{D})$.

References

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