Lyapunov Spectrum and Control Sets

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For my Parents

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Introduction

To understand the behavior of a control system, one has to analyze the controllability properties of the system. In this case the notion of control sets plays a major part. *Control sets* are maximal subsets of the state space of control systems, where complete controllability holds. They first appeared in the context of stochastic differential equations (cf. Arnold and Kliemann [2] (1987)). Control sets turned out to be an appropriate tool for analyzing control systems and for analyzing the qualitative behavior of nonlinear differential equations under bounded time-variant perturbations (cf. the book of Colonius and Kliemann [9]).

For control systems, the control sets and their domain of attraction can also be used to characterize the region, where the system is stabilizable (namely in the closure of the control sets). Nonlinear differential equations with bounded time-variant perturbations can be interpreted as control systems with bounded control range, and then the long-time behavior of the perturbed equations can be characterized by orbits, invariant sets and the domain of attraction of control sets, i.e. by control theoretic concepts.

In this thesis we consider nonlinear control affine systems with a singular point: A singular point is a point in the state space, where the right hand side of the control system vanishes for every applied control function. This means, that all trajectories which start in this singular point, stay for all time in this point.

We want to analyze under which conditions there are control sets near the singular point and under which conditions not. By linearization of the nonlinear system we get a bilinear control system. Knowledge about the behavior of the linearized system can be used to make local statements for the nonlinear system. This means you have to understand the bilinear system first. If we apply a control function on the bilinear control system, we get a linear nonautonomous differential equation. For analyzing the stability behavior of nonautonomous linear differential equation Lyapunov has introduced the so called *order numbers*, or (how we call them today in honor of their inventor) the *Lyapunov exponents* (cf. Lyapunov [23]). For autonomous linear systems, the eigenvalues of the right hand side determine the stability behavior of the system. For nonautonomous linear systems, the Lyapunov exponents are the appropriate generalization. Under some regularity conditions on the right hand side, the stability behavior of the nonlinear system is given by the Lyapunov exponents of its linearization.

All possible Lyapunov exponents which we get if we apply all admissible control functions on the bilinear control system form the *Lyapunov spectrum* The aim of the book is to use the Lyapunov spectrum for the characterization of the controllability properties of the nonlinear control system near a singular point. This means, we use the knowledge about the linearized system for a local analysis of the nonlinear system.

For showing the existence of control sets near a singular point it will turn out that it will suffice to know the existence of periodic control functions for which the linearized system has special Lyapunov exponents. Furthermore we will use the existence of (local) stable and unstable fibre bundles for the nonlinear (nonautonomous) differential equations.

The results of this book can also be interpreted as a kind of bifurcation theory of control sets near a singular point. By varying a parameter of the nonlinear control system (for example by changing the control range), the corresponding Lyapunov spectrum may vary, too. Because the existence of control sets near the singular point itself depends on the Lyapunov spectrum, it can happen that for certain parameters there are control sets near the singular point, but not for others (cf. Chapter 5.1).

This book should be read in a nonlinear way. The reader should start with the first chapter, then go directly to the appendix and then continue reading the book with Chapter 2. In the appendix he will be supplied with all technical devices of the theory of stable and unstable fibre bundles for nonlinear differential equations we will use in this book. I decided to put this chapter to the end of the book for two reasons. First, the theory of stable and unstable fibre bundles for nonautonomous differential equations is a theory for its own and therefore it should be separated from the control theoretic part of the book. Secondly the appendix should be used as a toolbox, where the necessary facts of this relatively new theory are packed together.

The *first chapter* is based on the book of Colonius and Kliemann [9]. We state the basic results on control sets and chain control sets for nonlinear control systems on Riemannian manifolds. Here the notion of the control flow plays an important role, because this enables us to use the theory of dynamical system for control systems. Then for a bilinear system, which we get if we linearize a nonlinear control system at a singular point, its projection on the real projective space is introduced. The control sets with nonvoid interior (the main control sets) are characterized by the eigenspaces of the bilinear system, if we apply piecewise constant and periodic control functions. These control functions are in relation with the system group. Several spectral concepts are introduced - the Lyapunov, the Floquet and the Morse spectrum.

The *second chapter* now considers nonlinear control systems with a singular point where the Lyapunov spectrum does not contain 0. We show, that if the Lyapunov spectrum has only positive values or only negative values, then there are no control sets near the singular point. We do not get this strong result, if we assume that the Lyapunov spectrum has positive and negative values. But we can show that there is a neighborhood around the singular point with the following property: Given two points of a control set which lie in this neighborhood, then one can steer one point to the other only by leaving this neighborhood at some time.

The *third chapter* supposes, that the linearized system has two periodic control functions. For the first control function the linearized system has only negative Lyapunov exponents and for the second control function, the Lyapunov spectrum has positive *and* negative values. We get the existence of such control functions for example if we assume, that the Lyapunov spectrum has 0 in its interior. This means we get a control set with nonvoid interior such that the singular point lies in the closure of the control set. We characterize the control set by unstable fibre bundles which we get if we apply the control function with the positive and negative Lyapunov exponents.

The *fourth chapter* deals with a special case of the third chapter. Here we assume, that there is a periodic control function such that the linearized system has only one positive Lyapunov exponent, and the others are all negative. In this case we can show uniqueness of the resulting control set. If we have two such control functions, then we get more than one control set and we investigate which of these control sets coincide. Finally we come back to the Lyapunov spectrum and show under additional conditions on the spectrum, that the control sets of the nonlinear system at the singular point are locally characterized by the control sets of the linearized system.

In the *fifth chapter* we will illustrate the theoretical results of the pervious chapters by numerical examples. Here the perturbed Duffing-van der Pol equation serves as a two-dimensional example for the results of Chapter 2 and 4. The perturbed Lorenz equation is a three-dimensional example for the results of Chapter 4.

Finally in the *Appendix* we cite the results about the invariant fibre bundles for (nonautonomous) differential equations. This is mainly based on the book of S.Siegmund [29]. After the linear theory, the global theory of invariant fibre bundles, asymptotic phases and the Hartman-Grobman Theorem are introduced. This global theory works for systems which fulfill special conditions (for example on their Lipschitz constants). We will generalize this for periodic systems with a fixed point, by reducing the given system in such a way, that we can apply the global theory. For the given system we will get local results near the singular point.

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Chapter 1

Linear and Nonlinear Control Systems

In this first chapter, we will introduce the notions of geometric control theory and bilinear control systems, which will accompany us throughout the following chapters.

In the first part, we consider control affine systems on Riemannian manifolds. For these quite general systems, the notion of control sets and chain control sets are defined. Control sets are the main objects of this book. These objects are in relation with the long time behavior of trajectories of the given control system. To analyze the long time behavior, we enlarge the nonlinear control system, to get a dynamical system, which is called the control flow. Now with the ω-limit sets of this dynamical system we can characterize the control sets. Of great importance for the Chapters 3 and 4 will be Proposition 1.1.21. It supplies us with an existence theorem for control sets, which can be used, if the control affine system has a singular point.

The second part then concentrates on bilinear control systems on $\mathbb{R}^d$. If we have a nonlinear control affine system with singular point and linearize this system at its singular point, then we get a bilinear control system. For analyzing the stability behavior of the nonlinear system near the singular point, one has to understand its linearization, the bilinear control system. For doing this, we look at the projection of the bilinear system on the real projective space. The control sets with nonvoid interior of the projected system then can be described by the eigenspaces of the bilinear control system, if we apply constant control functions.

Then we introduce several spectral concepts, the Floquet, the Morse and the Lyapunov spectrum for the bilinear system. The Floquet spectrum is a subset of the Lyapunov spectrum, and this again is a subset of the Morse Spectrum. We will give conditions, under which the closure of the Floquet and the Lyapunov and the Morse Spectrum coincide.

Most parts of this chapter are taken from the book of Colonius and Kliemann [9]. The Appendix A of [9], serves as quite a detailed introduction into geometric control theory. Other references on geometric control theory are the books of V.Jurdjevic [21], H.Neijmeijer and A.J. van der Schaft [24] and A.Isidori [20]. The facts about control sets, the control flow, ω-limit sets and the spectral theory can be found in the book of Colonius and Kliemann [9]. The items about bilinear control systems and its projection on projective space can also be found in the article [8].

1.1 Nonlinear Control Affine Systems

1.1.1 Control Sets and Chain Control Sets

We will introduce now the control sets for control affine systems on Riemannian manifolds. Later we will only consider control systems on $\mathbb{R}^d$, but because control sets are so important in this book, we introduce them here in a quite general environment.

In the following denote by M a connected Riemannian C^∞-manifold of dimension $d < \infty$. We denote its tangent bundle by $\mathbf{T}M$ and the tangent space of M at a point $x \in M$ by $\mathbf{T}_x M$.

We consider the following class of control affine systems on M

$$\begin{array}{l} \dot{x} = f_0(x) + \sum_{i=1}^m u_i(t) f_i(x) \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\}. \end{array} \tag{1.1}$$

We assume that $f_0, \dots, f_m$ are C^∞ vector fields on M and that U is a compact and convex subset of $\mathbb{R}^m$. Furthermore, suppose that for all $(u, x) \in \mathcal{U} \times M$ the equation (1.1) has a unique solution $\varphi(t, \tau, x, u)$ for all $t, \tau \in \mathbb{R}$, with $\varphi(\tau, \tau, x, u) = x$.

The set of points, which are reachable from a given point $x \in M$ and are controllable to x are defined by

Definition 1.1.1 *For $x \in \mathbb{R}^d$ the set of points reachable from x up to time $T > 0$ is*

$$\mathcal{O}^+_{\leq T}(x) := \{y \in M : \textit{there are } 0 \leq t \leq T \textit{ and } u \in \mathcal{U} \textit{ with } y = \varphi(t, 0, x, u)\}.$$

The set of points controllable to x within the time $T > 0$ is

$$\mathcal{O}^-_{\leq T}(x) := \{y \in M : \textit{there are } 0 \leq t \leq T \textit{ and } u \in \mathcal{U} \textit{ with } x = \varphi(t, 0, y, u)\}.$$

Furthermore let $\mathcal{O}^+(x) := \bigcup_{T>0} \mathcal{O}^+_{\leq T}(x)$ and $\mathcal{O}^-(x) := \bigcup_{T>0} \mathcal{O}^-_{\leq T}(x)$ denote the reachable set from x and the controllable set to x, respectively. We also call $\mathcal{O}^+(x)$ the forward *or* positive orbit *and $\mathcal{O}^-(x)$ the* backward *or* negative orbit *of x.*

Very often we will impose the following regularity condition on the control system (1.1):

Definition 1.1.2 *The control system (1.1) is called* locally accessible *from $x \in M$ if for every $T > 0$ the sets $\mathcal{O}^+_{\leq T}(x)$ and $\mathcal{O}^-_{\leq T}(x)$ have nonvoid interior.*

A thorough investigation of local accessibility is given in Appendix A of [9]. Local accessibility is guaranteed by the following *accessibility rank condition*: Let

$$\mathcal{LA} := \mathcal{LA}\left\{ f_0 + \sum_{i=1}^{m} u_i f_i : u \in U \right\} \tag{1.2}$$

denote the Lie algebra generated by the vector fields f_i and for $x \in M$ let $\Delta_{\mathcal{LA}}(x)$ be the subspace of the tangent space $\mathbf{T}_x M$ generated by the vector fields in $\mathcal{LA}$.

Definition 1.1.3 *The* accessibility rank condition *requires*

$$\dim \Delta_{\mathcal{LA}}(x) = \dim M = d \text{ for all } x \in M.$$

If the system (1.1) fulfills the accessibility rank condition, then it is locally accessible from every point $x \in M$. Hence local accessibility can be checked by computing Lie brackets. We will do this in Chapter 5 for the examples of the perturbed Duffing-van der Pol oscillator and the perturbed Lorenz equation.

Remark 1.1.4 *If M is a C^∞-manifold and f_i are C^∞-vector fields and the accessibility rank condition is fulfilled, then by the Theorem of Frobenius the system is* Lie-determined. *If M is an analytic manifold and the vector fields are analytic, then system (1.1) is also Lie-determined, by the Theorem of Nagano (cf. Appendix A in [9]). Thus the system restricted to a maximal integral manifold satisfies the accessibility rank condition and is locally accessible (cf Remark A.4.8 in [9]).*

After introducing the positive orbit, we now come to control sets, the objects that play the crucial role in this thesis.

Definition 1.1.5 *A set $D \subset M$ is called control set of the system (1.1) if*

(i) for all $x \in D$ one has $D \subset \operatorname{cl} \mathcal{O}^+(x)$,

(ii) for all $x \in D$ there is a control function $u \in \mathcal{U}$ such that $\varphi(t, 0, x, u) \in D$ for all $t \geq 0$,

(iii) D is maximal with the properties (i) and (ii), that is, if $D' \supset D$ satisfies the conditions (i) and (ii), then $D' = D$.

Property (i) means, that we have *approximative controllability* in D. For every $x, y \in D$ and every neighborhood N of y there exists a control function $u \in \mathcal{U}$ and a time $t \geq 0$ such that $\varphi(t, 0, x, u) \in N$. This property does not change if instead of locally integrable controls piecewise constant or piecewise continuous ones are used (cf. Appendix A in [9]). If the control set has nonvoid interior, then under local accessibility, we even have complete controllability in the interior of the control set.

Property (ii) means, that if we start in a point $x \in D$, we can stay for all positive times in D. Thus every control set is *viable* in the sense of J.P.Aubin [3]. This property is introduced to exclude trivial cases, since every one-point set in M satisfies the property (i).

Finally the maximum property (iii) is imposed for simplicity. Every set D_0 which satisfies the properties (i) and (ii) is contained in a maximal set of this type, i.e. a control set. In particular, control sets are pairwise disjoint.

For general systems, there exist control sets with void interior. But they are not important in this book, because all control sets, which are of interest for us, will have nonvoid interior.

An important class of control sets are the invariant control sets.

Definition 1.1.6 *A control set $C \subset \mathbb{R}^d$ is called* invariant *control set if* $\operatorname{cl} C = \operatorname{cl} \mathcal{O}^+(x)$ *for all $x \in C$. All other control sets are called* variant.

To get a global picture of the control affine system (1.1) and its control sets we also study the set of points which can be steered approximately to a control set D. This will be used in Section 4.2 and Theorem 4.2.5.

Definition 1.1.7 *The* domain of attraction *of a control set D is defined as*

$$\mathbf{A}(D) := \{x \in M : \operatorname{cl} \mathcal{O}^+(x) \cap D \neq \emptyset\}.$$

The reachability order $\preceq$ *on the control sets of the system (1.1) is given by*

$$D \preceq D' \text{ if } D \cap \mathbf{A}(D') \neq \emptyset.$$

If $D \preceq D'$ and $D \neq D'$ we also write $D \prec D'$.

In general, limit sets of controlled trajectories (for $t \to \pm\infty$) need not to be contained in control sets. See for example Proposition 1.1.21 below, where we have to impose some special conditions on the control system (1.1) to actually get the existence of a control set. However under some milder conditions, the limit sets are contained in chain control sets (cf. Section 4 in [9]). They will play also an important part for the linearized system in Section 1.2. First we have to introduce the notion of (ε, T)-chains, which can be interpreted as controllability allowing (small) jumps between pieces of trajectories.

Definition 1.1.8 *Fix $x, y \in M$ and let $\varepsilon, T > 0$. A* controlled (ε, T)-chain ζ *from x to y is given by $n \in \mathbb{N}, x_0, \ldots, x_n \in M, u_0, \ldots, u_{n-1} \in \mathcal{U}$ and $t_0, \ldots, t_{n-1} \geq T$ with $x_0 = x, x_n = y$ and*

$$d(\varphi(t_j, 0, x_j, u_j), x_{j+1}) \leq \varepsilon \text{ for all } j = 0, \ldots, n-1.$$

If for every $\varepsilon, T > 0$ there is an (ε, T)-chain from x to y, then the point x is chain controllable *to y.*

In analogy to control sets, chain control sets are defined as maximal regions of chain controllability.

Definition 1.1.9 *A set $E \subset M$ is called* chain control set *of system (1.1) if*

(i) for all $x, y \in E$ and all $\varepsilon, T > 0$ there is a controlled (ε, T)-chain from x to y,

(ii) for all $x \in E$ there is a $u \in \mathcal{U}$ such that $\varphi(t, 0, x, u) \in E$ for all $t \in \mathbb{R}$,

(iii) E is maximal with respect to set inclusion with the properties (i) and (ii).

Note that chain control sets are always closed and that for every control set D with nonvoid interior, there is a chain control set E with $D \subset E$ (see [9], Corollary 4.3.12).

1.1.2 Control Flow and Limit Sets

We want to study the control system (1.1) as a family of ordinary differential equations indexed by the functions $u \in \mathcal{U}$. Our main interest lies in the long time behavior of the trajectories $\varphi(t, 0, x, u)$. For studying this behavior, we will use concepts and techniques from the theory of dynamical system. But for a given $u \in \mathcal{U}$ the mapping $x \mapsto \varphi(t, 0, x, u)$ does not define a flow on M and it does not help us to handle the dependency on u. Thus we have to introduce the concept of the *control flow*, which will define a continuous dynamical system.

For this purpose we first have to introduce a dynamical system on $\mathcal{U}$, the shift θ on $\mathcal{U}$.

Definition 1.1.10 *The* shift $\theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}$ *is defined by*

$$\theta_t(u) := \theta(t, u) := u(t + \cdot).$$

We call the pair $(\mathcal{U}, \theta)$ the shift space.

Lemma 1.1.11 *The set $\mathcal{U}$ is compact and metrizable in the weak* topology of $L_\infty(\mathbb{R}, \mathbb{R}^m) = (L_1(\mathbb{R}, \mathbb{R}^m))^*$, a metric is given by*

$$d(u, v) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\left|\int_{\mathbb{R}} \langle u(t) - v(t), x_n(t)\rangle dt\right|}{1 + \left|\int_{\mathbb{R}} \langle u(t) - v(t), x_n(t)\rangle dt\right|},$$

where $(x_n)_{n \in \mathbb{N}}$ is a countable, dense subset of $L_1(\mathbb{R}, \mathbb{R}^m)$ and $\langle \cdot, \cdot \rangle$ denotes an inner product in $\mathbb{R}^m$. With this metric, $\mathcal{U}$ is a compact, complete, separable metric space.

Proof. Lemma 4.2.1 in [9]. ■

The weak* topology is the weakest topology such that for all $x \in L_1(\mathbb{R}, \mathbb{R}^m)$ the linear functional $u \to \int_{\mathbb{R}} \langle u(t), x(t)\rangle dt$ on $L_\infty(\mathbb{R}, \mathbb{R}^m)$ is continuous.

Lemma 1.1.12 *The shift θ defines a continuous dynamical system on M and is topologically mixing, topologically transitive, and chain transitive.*

Proof. Lemma 4.2.4 and Lemma 4.2.7 in [9]. ■

By combining the shift with the solution mapping we obtain the control flow.

Definition 1.1.13 *The* control flow *of the system (1.1) is defined by*

$$\begin{array}{rccc} \Phi: & \mathbb{R}\times\mathcal{U}\times M & \to & \mathcal{U}\times M \\ & (t,u,x) & \mapsto & (\theta_t(u),\varphi(t,0,x,u)). \end{array}$$

Lemma 1.1.14 *The map Φ defines a continuous dynamical system on $\mathcal{U}\times M$.*

Proof. Cf. Lemma 4.3.2 in [9]. ∎

The M-component satisfies the cocycle property

$$\varphi(t+\tau,0,x,u)=\varphi(t,0,\varphi(\tau,0,x,u),\theta_\tau(u)).$$

Hence Φ is a skew-product flow. Because $(\mathcal{U}\times M,\Phi)$ is a continuous dynamical system, we can define the ω-limit set of a pair $(u,x)\in\mathcal{U}\times M$.

Definition 1.1.15 *For the dynamical system $(\mathcal{U}\times M,\Phi)$ and a pair $(u,x)\in\mathcal{U}\times M$ the ω-limit set is defined by*

$$\omega(u,x):=\left\{(v,y)\in\mathcal{U}\times M: \begin{array}{l}\text{there is a sequence } (t_k)_{k\in\mathbb{N}}\in\mathbb{R} \text{ with } t_k\to\infty \\ \text{for } k\to\infty \text{ and } \lim_{k\to\infty}\Phi_{t_k}(u,x)=(v,y)\end{array}\right\}.$$

Corollary 1.1.16 *Let $(u,x)\in\mathcal{U}\times M$ such that the set $\{\varphi(t,0,x,u):t\geq 0\}$ is bounded. Then the ω-limit set $\omega(u,x)$ is nonempty, invariant and compact.*

Proof. Because the sequence $(\Phi_k(u,x))_{k\in\mathbb{N}}$ is bounded in $\mathcal{U}\times M$ it has a convergent subsequence. Thus the limit of this subsequence is an element of $\omega(u,x)$, which is therefore nonempty. Now note that $\{\Phi_t(u,x):t\geq\tau\}$ is bounded for all $\tau\geq 0$ and therefore $\mathrm{cl}\{\Phi_t(u,x):t\geq\tau\}$ is compact. Because we have $\omega(u,x)=\bigcap_{\tau\in[0,\infty)}\mathrm{cl}\{\Phi_t(u,x):t\geq\tau\}$, it follows that $\omega(u,x)$ is compact.

Finally for showing invariance, we have to show that for every $(v,y)\in\omega(u,x)$ we have $\{\Phi_\tau(v,y):\tau\in\mathbb{R}\}\subseteq\omega(u,x)$. Let $(v,y)\in\omega(u,x)$ and $(t_k)_{k\in\mathbb{N}}\subset[0,\infty)$ be a sequence which goes to infinity with $\lim_{k\to\infty}\Phi_{t_k}(u,x)=(v,y)$. Then the sequence $s_k:=t_k+\tau\in[\tau,\infty), k\in\mathbb{N}$ also goes to infinity and because of continuity of the flow Φ, we have

$$\lim_{k\to\infty}\Phi_{s_k}(u,x)=\Phi_\tau\circ\lim_{k\to\infty}\Phi_{t_k}(u,x)=\Phi_\tau(v,y).$$

This means, that $\Phi_\tau(v,y)$ is an element of $\omega(u,x)$. ∎

In order to describe the connection between ω-limit sets and control sets, we have to introduce the following terms.

Definition 1.1.17 *A pair $(u,x)\in\mathcal{U}\times M$ is called* inner pair , *if there exists a $T>0$ such that $\varphi(T,0,x,u)\in\mathrm{int}\,\mathcal{O}^+(x)$. The pair (u,x) is called* strong inner pair *if for all $t\in\mathbb{R}$ we have $\varphi(t,0,x,u)\in\mathrm{int}\,\mathcal{O}^+(x)$.*

In general, it is quite difficult to check if a given $(u, x) \in \mathcal{U} \times M$ is an inner pair or a strong inner pair. For constant controls, however, one can give a criterion in terms of Lie brackets. If we denote by $[X, Y]$ the Lie bracket of two vector fields X and Y, we define

$$\begin{aligned} \mathrm{ad}_X^0\, Y &= Y \\ \mathrm{ad}_X^1\, Y &= [X, Y], \\ \mathrm{ad}_X^k\, Y &= ad_X(ad_X^{k-1} Y) \text{ for } k = 2, 3, \dots . \end{aligned}$$

Proposition 1.1.18 *Let $u \in \mathrm{int}\, U$ be a constant control and fix a point $x \in M$. Write $f = f_0 + \sum_{i=1}^m u_i f_i$ and assume that there is $T > 0$ such that at $y = \varphi(T, 0, x, u)$ the controllability rank condition*

$$\mathrm{span}\{(f(y), ad_f^k f_i)(y) : i = 1, \dots, m, k = 0, 1, \dots\} = \mathbf{T}_y M \tag{1.3}$$

holds. Then for every $(v, z) \in \mathcal{U} \times M$ with $v_{[0,T]}$ in a neighborhood of $u \in L_\infty([0, T], \mathbb{R}^m)$ and a z in a neighborhood of $x \in M$, it follows, that $\varphi(T, z, v) \in \mathrm{int}\, \mathcal{O}_T^+(z)$ and in particular, (v, z) is an inner pair.

Proof. Proposition 4.5.19 in [9]. ■

Corollary 1.1.19 *If under the condition of Proposition 1.1.18 for every $T > 0$ the controllability rank condition (1.3) is fulfilled in x, then (u, x) is a strong inner pair.*

Proof. By Proposition 1.1.18 it follows, that $\varphi(T, x, u) \in \mathrm{int}\, \mathcal{O}_T^+(x)$ for all $T > 0$. ■

The following corollary shows, how the ω-limit sets of trajectories are in relation with control sets having nonvoid interior. The projection π_M of $\mathcal{U} \times M$ onto M is defined by

$$\pi_M(u, x) := x.$$

Corollary 1.1.20 *Consider a pair $(u, x) \in \mathcal{U} \times M$ such that $\{\varphi(t, 0, x, u) : t \geq 0\}$ is bounded and suppose that the accessibility rank condition holds on $\pi_M \omega(u, x)$. Then the following assertions are equivalent:*

(a) There is a control set D with $\pi_M \omega(u, x) \subset \mathrm{int}\, D$.

(b) The set $\omega(u, x)$ consists of inner pairs.

Proof. See Corollary 4.5.9 in [9]. ■

In the sections which follow, we will look at control systems with a singular point, i.e. control systems, which have the property, that there is a $x^* \in M$ such that $f_i(x^*) = 0$. This means, that $\varphi(t, 0, x, u) = x$ for all $t \in \mathbb{R}$ and all $x \in M$. Therefore the system is *not* locally accessible at the singular point. Thus if we have a pair $(u, x) \in \mathcal{U} \times M$ such that $x^* \in \pi_M \omega(u, x)$ we can not apply Corollary 1.1.20. The following proposition will show, that under additional assumptions on the ω-limit set, we nevertheless can have an existence criterion for control sets, which have nonvoid intersection with the ω-limit set.

Proposition 1.1.21 *Consider the nonlinear control system (1.1). Assume that the control system is Lie-determined and has a maximal integral manifold J with maximal dimension. Let $x \in J$ and $u \in \mathcal{U}$, such that $\{\varphi(t,0,x,u) : t \geq 0\}$ is bounded . Suppose further that there is a compactum $\mathcal{K} \subset \mathcal{U} \times J$ such that the following properties are satisfied.*

(1) $\mathcal{K} \cap \omega(u,x) \neq \emptyset$.

(2) For all $(v,y) \in \mathcal{K}$ there is a $t > 0$ such that $\varphi(t,0,y,v) \in \operatorname{int} \mathcal{O}^+(y)$ (inner pair condition).

(3) There is a $s^* > 0$ such that for all $(v,y) \in \Phi_{-s^*}\mathcal{K}$ and all $t > 0$ we have $\varphi(t,y,v) \in \operatorname{int} \mathcal{O}^+(y)$ (strong inner pair condition).

Then there exists a control set $D \subset \operatorname{cl} J$ with

$$\pi_M(\mathcal{K} \cap \omega(u,x)) \subset \operatorname{int} D. \tag{1.4}$$

Proof. Since the system is Lie-determined, its restriction on J is locally accessible (cf. Remark A.4.8 in [9]). Now let $(v,y),(w,z) \in \mathcal{K} \cap \omega(u,x)$. (The intersection is nonempty due to the assumptions). We show, that $z \in \operatorname{int} \mathcal{O}^+_{\leq\tau}(y)$ for a $\tau > 0$.

Because of $(v,y) \in \mathcal{K}$ for $T_0 > 0$ there is by *(2)* an $\varepsilon_0, S_0 > 0$ with

$$B_{\varepsilon_0}(\varphi(T_0,0,y,v)) \subset \operatorname{int} \mathcal{O}^+_{\leq T_0+S_0}(y) \tag{1.5}$$

Because $(v,y) \in \omega(u,x)$ there is a $t_0 > 0$ with

$$\varphi(t_0,0,x,u) \in B_{\varepsilon_0}(\varphi(T_0,0,y,v)) \subset \operatorname{int} \mathcal{O}^+_{\leq T_0+S_0}(y) \tag{1.6}$$

From *(3)* it follows, that for every $T > 0$ there is a neighborhood $\mathcal{N}_J(T)$ of $\Phi_{-s^*}\mathcal{K}$ in $\mathcal{U}\times J$ and $\varepsilon_1 := \varepsilon(T), S_1 := S(T) > 0$ with

$$B_{\varepsilon_1}(\varphi(T,0,p,a)) \subset \operatorname{int} \mathcal{O}^+_{\leq T_1+S_1}(p) \text{ for all } (a,p) \in \mathcal{N}_J(T) \tag{1.7}$$

(cf. Remark 4.5.6 in [9]). Now by choosing $T_1 := s^*$ we get a neighborhood $\mathcal{N} := \mathcal{N}_J(T_1)$ of $\Phi_{-T_1}\mathcal{K}$ and $\varepsilon_1 := \varepsilon(T_1), S_1 := S(T_1) > 0$ with

$$B_{\varepsilon_1}(\varphi(T_1,0,p,a)) \subset \operatorname{int} \mathcal{O}^+_{\leq T_1+S_1}(p) \text{ for all } (a,p) \in \mathcal{N}.$$

Note that $(\theta_{-T_1}w, \varphi(-T_1,0,z,w)) \in \Phi_{-T_1}(\mathcal{K})$ and therefore

$$(\theta_{-T_1}w, \varphi(-T_1,0,z,w)) \in \mathcal{N}.$$

The topology on $\mathcal{U}\times J$ is the induced product topology from $\mathcal{U} \times M$, thus there are open neighborhoods $V \subset J$ of $\varphi(-T_1,0,z,w)$ and $W \subset \mathcal{U}$ of $\theta_{-T_1}w$ with $W \times V \subset \mathcal{N}$. Because of continuous dependency on init values, we can choose V small enough, such that

$$\varphi(T_1,0,V,\theta_{-T_1}w) \subset B_{\frac{\varepsilon_1}{2}}(z). \tag{1.8}$$

Since $(w, z) \in \omega(u, x)$ and because of invariance of $\omega(u, x)$ we have

$$(\theta_{-T_1} w, \varphi(-T_1, 0, z, w)) \in \omega(u, x).$$

Thus there is a time $t_1 > t_0$ with $\varphi(t_1, 0, x, u) \in V$, and it follows that

$$(\theta_{-T_1} w, \varphi(t_1, 0, x, u)) \in W \times V \subset \mathcal{N}.$$

This means, that

$$B_{\varepsilon_1}(\varphi(T_1, 0, \varphi(t_1, 0, x, u), \theta_{-T_1} w)) \subset \operatorname{int} \mathcal{O}^+_{\leq T_1 + S_1}(\varphi(t_1, 0, x, u)).$$

Now define the control function $\tilde{u} : \mathbb{R} \to U$ by

$$\tilde{u}(t) := \begin{cases} u(t) & \text{for } t < t_1, \\ w(t - T_1 - t_1) & \text{for } t \geq t_1. \end{cases}$$

Together with (1.8) we get

$$\begin{aligned} z &\in B_{\varepsilon_1}(\varphi(t_1 + T, 0, x, \tilde{u})) \\ &\subset \operatorname{int} \mathcal{O}^+_{\leq T_1 + S_1}(\varphi(t_1, 0, x, u)) \\ &= \operatorname{int} \mathcal{O}^+_{\leq T_1 + S_1}(\varphi(t_1 - t_0, 0, \varphi(t_0, 0, x, u), u(t_0 + \cdot))) \\ &\subset \operatorname{int} \mathcal{O}^+_{\leq T_1 + S_1 + t_1 - t_0}(\varphi(t_0, 0, x, u)) \\ &\subset \operatorname{int} \mathcal{O}_{\leq T_0 + S_0 + T_1 + S_1 + t_1 - t_0}(y) \end{aligned}$$

where the last line follows from (1.6). Thus we have shown, that there exists a $\tau > 0$ with $z \in \operatorname{int} \mathcal{O}^+_{\leq \tau}(y)$. Vice versa, one can show, that $y \in \operatorname{int} \mathcal{O}^+_{\leq \tilde{\tau}}(z)$ for a $\tilde{\tau} > 0$. Now there exists a neighborhood V_z of z (relative to J) with $V_z \subset \operatorname{int} \mathcal{O}^+(y)$. By local accessibility on J the set $V := \operatorname{int} \mathcal{O}^-(z) \cap V_z$ is nonvoid. Every $p \in V$ can be reached from y and hence from z and therefore from every point in $\operatorname{int} \mathcal{O}^-(z)$. Thus $V \subset D$ for some control set D. Finally, since $z \in D$ can be reached in finite time from $\operatorname{int} V \subset \operatorname{int} D$, it follows that $z \in \operatorname{int} D$. Thus we have shown $\pi_M(\mathcal{K} \cap \omega(u, x)) \subset \operatorname{int} D$. Furthermore, Proposition 3.2.19 in [9] implies $D \subset \operatorname{cl} J$. ■

1.2 Bilinear Control Systems

1.2.1 Main Control Sets and Systems Group

The stability behavior of bilinear control systems can be characterized by Lyapunov exponents (cf. for example Chapter 2.1). For studying the Lyapunov exponents the projection of the bilinear system on projective space plays an important part. We will analyze the control sets of the projected system. It turns out that the control sets with nonvoid interior can be described by the eigenspaces corresponding to piecewise constant and periodic control functions.

In the following chapters, a nonlinear control affine system on $\mathbb{R}^d$ of the form

$$\begin{array}{l} \dot{x} = f_0(x) + \sum_{i=1}^{m} u_i(t) f_i(x) \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\} \end{array} \tag{1.9}$$

is given. We assume that $f_0, \dots, f_m$ are C^2 vector fields on M and that U is a compact and convex subset of $\mathbb{R}^m$ containing 0. Furthermore, suppose that for all $(u, x) \in \mathcal{U} \times \mathbb{R}^d$ the equation (1.9) has a unique solution $\varphi(t, \tau, x, u), t, \tau \in \mathbb{R}$, with $\varphi(\tau, \tau, x, u) = x$.

As the title of this book indicates, we consider a special class of control systems, namely those with a singular point.

Definition 1.2.1 *The nonlinear system (1.9) has a* singular point $x^* \in \mathbb{R}^d$ *if*

$$f_i(x^*) = 0 \textit{ for all } i = 0, \dots, m.$$

It follows, that $\varphi(t, \tau, x^*, u) = x^*$ for all $t, \tau \in \mathbb{R}, u \in \mathcal{U}$. Now let us suppose, that the nonlinear system (1.9) has a singular point $x^* \in \mathbb{R}^d$. If we linearize the nonlinear system (1.9) at the singular point x^* we obtain the following bilinear control system on $\mathbb{R}^d$:

$$\begin{array}{l} \dot{x} = A_0 x + \sum_{i=1}^{m} u_i(t) A_i x \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m : u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\}, \end{array} \tag{1.10}$$

where $A_i := \left.\frac{\partial f_i}{\partial x}\right|_{x=x^*}$. We denote by the mapping $\mathbb{R} \times \mathbb{R} \to gl(\mathbb{R}^d)$, $(t, \tau) \longmapsto \eta(t, \tau, u)$ the fundamental solution of the linear system (1.10) for a $u \in \mathcal{U}$.

Associated with the bilinear control system is its projection on the real projective space $\mathbb{P}^{d-1}$:

$$\begin{array}{l} \dot{p} = h(p, u(t)) = h_0(p) + \sum_{i=1}^{m} u_i(t) h_i(p) \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\}, \end{array} \tag{1.11}$$

where

$$h_i(p) := (A_i - p^T A_i p \cdot \mathrm{id})p \text{ for } i = 0, \dots, m.$$

and ^{T} denotes the transposition and id the $d \times d$ identity matrix. We denote the solutions of (1.11) by $\mathbb{P}\eta(t, \tau, u)p$ with $\mathbb{P}\eta(\tau, \tau, u)p = p$ for $\tau, t \in \mathbb{R}, p \in \mathbb{P}^{d-1}, u \in \mathcal{U}$. Note, that this is a *nonlinear* control system on $\mathbb{P}^{d-1}$.

For a point $x \in \mathbb{R}^d \setminus \{0\}$ we denote its projection on $\mathbb{P}^{d-1}$ by $\mathbb{P}x$, and for a point $\mathbb{P}x \in \mathbb{P}^{d-1}$ we denote $\mathbb{P}^{-1}(\mathbb{P}x) := \{\alpha x : \alpha \in \mathbb{R}, \alpha \neq 0\}$. Then we have $\mathbb{P}(\eta(t, \tau, u)x) = \mathbb{P}\eta(t, \tau, u)\mathbb{P}x$ for all $t, \tau \in \mathbb{R}, x \in \mathbb{R}^d \setminus \{0\}$.

Very often in this section, we suppose that the Lie algebra rank condition is fulfilled.

Condition 1.2.2 *The projected control system (1.11) is locally accessible, which is equivalent to the Lie algebra rank condition*

$$\dim \mathcal{LA}\{h(\cdot, u) : u \in U\}(p) = d - 1 \textit{ for all } p \in \mathbb{P}^{d-1}. \tag{1.12}$$

Equivalence of local accessibility and the Lie algebra rank condition comes from the fact, that the projected system (1.11) is real analytic.

Next we define the systems group and the systems semigroup. First we define for abbreviation the set N

$$N := \left\{ A_0 + \sum_{i=1}^{m} u_i A_i : u \in U \right\} \subset gl(d, \mathbb{R}),$$

which consists of all right hand sides of the bilinear system (1.10), if we apply constant controls.

Definition 1.2.3 *The system group $\mathcal{G}$ and the semigroup $\mathcal{S}$ are defined by*

$$\mathcal{G} = \{\exp t_n B_n \cdot \ldots \exp t_1 B_1 : t_j \in \mathbb{R}, B_j \in N, j = 1, \ldots, n, n \in \mathbb{N}\},$$
$$\mathcal{S} = \{\exp t_n B_n \cdot \ldots \exp t_1 B_1 : t_j \geq 0, B_j \in N, j = 1, \ldots, n, n \in \mathbb{N}\}.$$

These groups correspond to fundamental solutions of the bilinear system (1.10) with piecewise constant controls. For $g \in \mathcal{S}$ there is a corresponding piecewise constant and periodic control function $u_g \in \mathcal{U}$. If $g = \exp t_n B_n \cdot \ldots \cdot \exp t_1 B_1$ with $B_j = A_0 + \sum_{i=1}^m u_i A_i, u_j \in U$ for $j = 1, \ldots, n$, then define

$$u_g(t) = u_j \text{ for } t \in \left[\sum_{i=0}^{j-1} t_j, \sum_{i=0}^{j} t_i\right) \text{ with } t_0 = 0,$$

and continue it Θ-periodically with $\Theta = \sum_{i=1}^n t_i$. It follows, that we have $\eta(\Theta, 0, u_g) = g$. Vice versa, for a piecewise constant control function $u : [0, t] \to U$ there exists a unique element $g_u \in \mathcal{S}$, defined by $g_u := \eta(t, 0, u)$.

The subsets of the group $\mathcal{G}$ and the semigroup $\mathcal{S}$ for which we have in the definition $\sum_{j=1}^n |t_j| = t$ are denoted by $\mathcal{G}_t$ and $\mathcal{S}_t$, and the subsets where $\sum_{j=1}^n |t_j| < t$ are denoted by $\mathcal{G}_{\leq t}$ and $\mathcal{S}_{\leq t}$.

The system group $\mathcal{G}$ and the semigroup $\mathcal{S}$ act by a natural way on $\mathbb{R}^d$, by gx for $g \in \mathcal{G}, x \in \mathbb{R}^d$. The Lie group $\mathcal{G}$ induces a Lie group $\mathbb{P}\mathcal{G}$ obtained via a projection denoted again by $\mathbb{P}$, identifying g_1 and g_2 if $g_1 = \alpha g_2$ for some $\alpha \neq 0$. This group $\mathbb{P}\mathcal{G}$ and the semigroup $\mathbb{P}\mathcal{S}$ correspond to the projective control system (1.11). The Lie algebra rank condition (1.12) implies that $\mathbb{P}\mathcal{G}$ acts transitively on $\mathbb{P}^{d-1}$ and the interior $\mathrm{int}\mathbb{P}\mathcal{S}_{\leq t}$ of $\mathbb{P}\mathcal{S}_{\leq t}$ relative to $\mathbb{P}\mathcal{G}$ is nonvoid for every $t > 0$.

For notational convenience we write in the following $g \in \mathrm{int}\mathcal{S}_{\leq t}$ if we mean elements of $g \in \mathcal{S}$ with $\mathbb{P}g \in \mathrm{int}\,\mathbb{P}\mathcal{S}_{\leq t}$.

The projective system (1.11) is a nonlinear control system on the compact space $\mathbb{P}^{d-1}$. But because it is induced by the linear system (1.10) we can make a quite sharp characterization of the control sets with nonvoid interior of the projective system. Before doing this, we have to introduce some more notations.

For an element $g \in \mathcal{S}$ denote by $\operatorname{spec}(g) \subset \mathbb{C}$ the *spectrum* of the matrix $g \in \mathcal{S}$. For a $\sigma \in \operatorname{spec}(g)$ we denote by

$$E(\sigma) \subset \mathbb{R}^d$$

the *generalized real eigenspace* corresponding to σ. The projection of $E(\sigma)$ onto $\mathbb{P}^{d-1}$ is defined by

$$\mathbb{P}E(\sigma) := \{\mathbb{P}x : x \in E(\sigma), x \neq 0\}.$$

Theorem 1.2.4 *Consider the projected system (1.11) and assume that the local accessibility condition 1.2.2 holds.*

(a) The system has k control sets D_i with nonvoid interior in $\mathbb{P}^{d-1}$ and $1 \leq k \leq d$; we call these control sets the main control sets *of the system (1.11).*

(b) The main control sets are linearly ordered, with $D_i \prec D_j$ if there exists an $x \in D_i$ with $\mathcal{O}^+(x) \cap D_j \neq \emptyset$. We use the enumeration $D_1 \prec \ldots \prec D_k$. The maximal control set $C := D_k$ is invariant and closed, the minimal control set $C^- := D_1$ is open.

(c) For every $t > 0$, every $g \in \operatorname{int} \mathcal{S}_{\leq t}$ and every $\sigma \in \operatorname{spec}(g)$ there is a main control set D_i such that $E(\sigma)$ satisfies $\mathbb{P}E(\sigma) \subset \operatorname{int} D_i$. Vice versa, each $x \in \operatorname{int} D_i$ is an eigenvector for a real eigenvalue of some $g \in \operatorname{int} \mathcal{S}_t$ for some $t > 0$.

(d) For every $g \in \mathcal{S}$ and every $\sigma \in \operatorname{spec}(g)$ there is a main control set D_i with $\mathbb{P}E(\sigma) \cap \operatorname{cl} D_i \neq \emptyset$. Vice versa, for every D_i and every $g \in \mathcal{S}$ there exists $\sigma \in \operatorname{spec}(g)$ with $\mathbb{P}E(\sigma) \cap \operatorname{cl} D_i \neq \emptyset$.

Proof. Theorem 7.1.1. and Theorem 7.3.3. in [9]. ■

This theorem characterizes the interior of the control set with nonvoid interior on $\mathbb{P}^{d-1}$ via the eigenspaces of elements in $\operatorname{int} \mathcal{S}$. But for a general $g \in \mathcal{S}$ (which does not lie in the interior of $\mathcal{S}$) the eigenspaces need not to be contained in the closure of the control sets with nonvoid interior, a counterexample is given by Example 7.3.14 in [9].

For $g \in \mathcal{S}$ and a main control set D we define

$$D(u_g) = \{x \in \mathbb{P}^{d-1} : \mathbb{P}\eta(t, 0, u_g) \in \operatorname{cl} D \text{ for all } t \in \mathbb{R}\}.$$

For $g \in \operatorname{int} \mathcal{S}_{\leq t}$ and each main control set D we have

$$D(u_g) = \mathbb{P}\Big(\bigoplus_{\sigma} E(\sigma)\Big),$$

where the sum is taken over all $\sigma \in \operatorname{spec}(g)$ with $\mathbb{P}E(\sigma) \subset \operatorname{int} D$. We get

$$\mathbb{R}^d = \bigoplus_{i=1}^{k} \Big(\bigoplus_{\sigma} E(\sigma)\Big) = \bigoplus_{i=1}^{k} \mathbb{P}^{d-1}[D_i(u_g)], \tag{1.13}$$

see Theorem 7.3.10 and Remark 7.3.11 in [9].

Recall that for all main control sets D_i there exists a unique chain control set E with $D_i \subset E$. For general control affine systems, it can happen, that there are chain control sets, which do not contain any control set with nonvoid interior. However the next theorem shows, that for the projective system (1.11) in every chain control set there must be a main control set. Furthermore one chain control set can contain more than one main control set. An example for this is given in [9], Example 7.3.14.

Theorem 1.2.5 *Suppose, that the projected system (1.11) on $\mathbb{P}^{d-1}$ fulfills the accessibility rank condition (1.12). Then every chain control set E of the projected system contains a main control set . In particular, if k denotes the number of main control sets, and l the number of chain control sets , then $1 \leq l \leq k \leq d$.*

Proof. See Theorem 7.3.15 in [9]. ■

Associated with the bilinear system (1.10) is its *control flow* $\mathbf{T}\Phi : \mathbb{R} \times \mathcal{U}\times\mathbb{R}^d \to \mathcal{U}\times\mathbb{R}^d$, defined by

$$\mathbf{T}\Phi_t(u, x) = (\theta_t u, \eta(t, 0, u)x) \tag{1.14}$$

for $(t, u, x) \in \mathbb{R} \times \mathcal{U}\times\mathbb{R}^d$.

Remark 1.2.6 *We denote the control flow of the linearized system (1.10) not only by $\mathbf{T}\Phi$ to distinguish it from the control flow Φ of the nonlinear system (1.9). The reason is, that $\mathbf{T}\Phi$ is the linearization of the flow Φ, if we identify the tangent space of M in x^* with $\mathbb{R}^d$.*

For the projected control system (1.11) the *projected control flow* $\mathbb{P}\Phi$ is defined by

$$\begin{array}{rccc} \mathbb{P}\Phi : & \mathbb{R} \times \mathcal{U}\times\mathbb{P}^{-1} & \to & \mathcal{U}\times\mathbb{P}^{-1} \\ & (t, u, x) & \mapsto & (\theta_t u, \mathbb{P}\eta(t, 0, u)p). \end{array}$$

If $E \subset \mathbb{P}^{d-1}$ is a chain control set of the projected system (1.11), then its *lift* $\mathcal{E}$ is defined by

$$\mathcal{E} := \{(u, p) \in \mathcal{U} \times \mathbb{P}^{d-1} : \mathbb{P}\eta(t, 0, u)p \in E \text{ for all } t \in \mathbb{R}\}.$$

The space $\mathcal{U}\times\mathbb{R}^d$ can be viewed as a (trivial) vector bundle $\pi : \mathcal{U} \times \mathbb{R}^d \to \mathcal{U}$ by defining

$$\pi(u, x) = x.$$

Then the space $\mathcal{U} \times \mathbb{P}^{d-1}$ can be interpreted as the projective bundle $\mathbb{P}\pi : \mathcal{U} \times \mathbb{P}^{d-1} \to \mathcal{U}$ associated to the vector bundles $\pi : \mathcal{U}\times\mathbb{R}^d \to \mathcal{U}$ with $\mathbb{P}\pi(u, p) = p$. For more information about vector bundles, see [9] Appendix B, D.Husemoller [19] and M.Karoubi [22]. The control flow Φ defines a *linear* flow on the vector bundle $\pi : \mathcal{U} \times \mathbb{R}^d \to \mathcal{U}$ and $\mathbb{P}\Phi$ is its associated projected flow on $\mathbb{P}\pi : \mathcal{U} \times \mathbb{P}^{d-1} \to \mathcal{U}$, cf. [9].

To get a deeper insight into the relation of chain control sets and the main control sets and to analyze the Lyapunov spectrum which we define later, we must have a dynamical analysis of the control flow. This will be done in terms of lifts of chain control sets of the projected control flows $\mathbb{P}\Phi$ and exponentially separated subbundles of the control flow Φ of the bilinear system (1.10).

First we define the (fibrewise-) norm $|\cdot|$ on $\mathcal{U} \times \mathbb{R}^d$ by

$$|(u,x)| := \|x\| \text{ for every } (u,x) \in \mathcal{U} \times \mathbb{R}^d.$$

A pair of $\mathbf{T}\Phi$-invariant subbundles $(\mathcal{V}^+, \mathcal{V}^-)$ with $\mathcal{U}\times\mathbb{R} = \mathcal{V}^+ \oplus \mathcal{V}^-$ (Whitneysum) is called *exponentially separated* if there are $c > 1$ and $\mu > 0$ with

$$|\mathbf{T}\Phi_t(u,x^+)| \le ce^{-\mu t}|\mathbf{T}\Phi_t(u,x^-)|$$

or in other words

$$\left\|\eta(t,0,u)x^+\right\| \le ce^{-\mu t}\left\|\eta(t,0,u)x^-\right\|$$

for all $t \ge 0$ and $(u,x^+) \in \mathcal{V}^+, (u,x^-) \in \mathcal{V}^-$ with $\|x^+\|, \|x^-\| = 1$. Hence exponential separation means that the exponential growth rate for a solution in the first bundle is uniformly smaller than for one in the second bundle if we start in the same fibre.

Theorem 1.2.7 *Let $E_1, \dots, E_l$ be the chain control sets of the projected system (1.11) and let $\mathcal{E}_i$ denote the lift of the chain control set $E_i, i = 1, \dots, l$. Then every $\mathcal{E}_i$ defines an invariant subbundle $\mathcal{V}_i$ of $\mathcal{U} \times \mathbb{R}^d$ via*

$$\mathcal{V}_i = \mathbb{P}^{-1}(\mathcal{E}_i) = \{(u,x) \in \mathcal{U} \times \mathbb{R}^d : x \neq 0 \textit{ implies } (u, \mathbb{P}x) \in \mathcal{E}_i\}$$

and the following decomposition in a Whitney sum holds:

$$\mathcal{V} = \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_l.$$

If $(\mathcal{V}^+, \mathcal{V}^-)$ are nontrivial exponentially separated subbundles, then there is $1 \le j < l$ such that

$$\mathcal{V}^+ = \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_j \textit{ and } \mathcal{V}^- = \mathcal{V}_{j+1} \oplus \ldots \oplus \mathcal{V}_l.$$

Conversely, subbundles $\mathcal{V}^+$ and $\mathcal{V}^-$ defined in this way are exponentially separated.

Proof. By Theorem 4.3.11 in [9] the lifts of the chain control sets are the maximal invariant chain transitive sets for the projected control flow $(\mathcal{U} \times \mathbb{P}^{d-1}, \mathbb{P}\Phi)$. Thus the Theorem follows from Theorem 5.2.5 and Corollary 5.2.11 in [9]. ■

Now we can characterize the exponentially separated subbundles of $\mathcal{U}\times\mathbb{R}^d$ by the chain control sets of the projected system.

Theorem 1.2.8 *Consider the bilinear control system (1.10) and assume that the Lie algebra rank condition (1.2.2) holds for the projected system (1.11). Then we have for each* $j = 1, \dots, l$

$$\mathcal{V}_j = \operatorname{cl}\left\{ (u_g, x) \in \mathcal{U} \times \mathbb{R}^d : \begin{array}{l} g \in int\mathcal{S}_{\leq t} \text{ for some } t > 0 \text{ and} \\ x \in \bigoplus \mathbb{P}^{-1}[D_i(u_g)] \text{ with summation} \\ \text{over all } i \text{ such that } D_i \subset E_j \end{array} \right\}.$$

In particular, for every $u \in \mathcal{U}$ *and every* $j = 1, \dots, l$ *there are* $x_1, \dots, x_{i_j} \in \mathbb{R}^d$ *such that*

$$\mathcal{V}_j(u) = span\{x_1, \dots, x_{i_j}\}$$

and every $\mathbb{P}x_i$ *lies in the closure of some main control set.*

Proof. See Theorem 7.1.3 in [9]. ■

The equation (1.13) and the Theorem 1.2.8 allows us to define the multiplicities of a main control set D and of a chain control set E as

$$\begin{array}{ll} m(D) := \#\{\sigma \in \operatorname{spec}(g) : \mathbb{P}E(\sigma) \subset \operatorname{int}D\} & \text{for all } g \in \operatorname{int}\mathcal{S}_{\leq t}, t > 0, \\ m(E) := \dim \mathcal{V}(u) & \text{for all } u \in \mathcal{U}, \end{array}$$

where each $\sigma \in \operatorname{spec}(g)$ is counted according to its multiplicity. $m(D)$ and $m(E)$ are well defined, because they are independent of $g \in \operatorname{int}\mathcal{S}_{\leq t}, t > 0$ and $u \in \mathcal{U}$. In particular if we denote by $D_{i_1}, \dots, D_{i_n}$ all those main control sets which are subsets of E_j we get

$$m(E_j) = m(D_{i_1}) + \ldots + m(D_{i_j}).$$

1.2.2 Floquet, Lyapunov and Morse Spectrum

We will now characterize the Lyapunov exponents of the bilinear control system (1.10). It will turn out, that the Lyapunov exponents are related to two other spectral concepts, the Floquet and the Morse exponents. Here the main and the chain control sets of the projected system (1.11) play an important role.

Now we introduce the spectral notions we will need in this and the following sections.

- The *Lyapunov exponent* of a solution $\eta(\cdot, 0, u)x$ of the bilinear system (1.10) for $x \neq 0$ is defined as

$$\lambda(u, x) := \limsup_{t\to\infty} \tfrac{1}{t} \log \|\eta(t, 0, u)x\|.$$

- The *Floquet spectrum over a main control set* D of the projected system (1.11) is

$$\Sigma_{Fl}(D) := \left\{ \lambda(u, x) : \begin{array}{l} x \in \operatorname{int}D, \ u \text{ is piecewise constant and} \\ \tau\text{-periodic for some } \tau \geq 0 \text{ with } \mathbb{P}\eta(\tau, 0, u)x = x \end{array} \right\}.$$

- The *Floquet spectrum* of the bilinear system (1.10) is defined by

$$\Sigma_{Fl} = \bigcup_{i=1}^{k} \Sigma_{Fl}(D_i).$$

- The *Lyapunov spectrum over a main control set* D of the projected system (1.10) is defined by

$$\Sigma_{Ly}(D) := \left\{\lambda(u,x) : (u,x) \in \mathcal{U} \times \mathbb{P}^{d-1}, \mathbb{P}\eta(t,0,u)x \in \mathrm{cl}D \text{ for all } t \geq 0\right\}.$$

- The *Lyapunov spectrum over a subbundle* $\mathcal{V} \subset \mathcal{U} \times \mathbb{R}^d$ is defined by

$$\Sigma_{Ly}(\mathcal{V}) := \{\lambda(u,x) : (u,x) \in \mathcal{V}, x \neq 0\}$$

- The *Lyapunov spectrum* of the bilinear system (1.10) is defined by

$$\Sigma_{Ly} := \{\lambda(u,x) : (u,x) \in \mathcal{U} \times \mathbb{R}^d, x \neq 0\}.$$

- The *finite time exponential growth rate* or the *chain exponent* of an (ε, T)-chain ζ in $\mathbb{P}^{d-1}$, given by $n \in \mathbb{N}, t_0, \ldots, t_n \geq T$, and $(u_0, p_0), \ldots, (u_n, p_n) \in \mathcal{U} \times \mathbb{P}^{d-1}$ is defined by

$$\lambda(\zeta) := \left(\textstyle\sum_{i=0}^{n-1} t_i\right)^{-1} \textstyle\sum_{i=0}^{n-1} (\log \|\mathbb{P}\eta(t_i,0,u_i)x_i\| - \log\|x_i\|)$$

where $x_i \in \mathbb{P}^{-1}(p_i)$.

- The *Morse spectrum over a chain control set* E of the projected system (1.11) is defined by

$$\Sigma_{Mo}(E) := \left\{ \lambda \in \mathbb{R} : \begin{array}{l} \text{there exist } \varepsilon^k \to 0, T^k \to \infty \text{ and} \\ (\varepsilon^k, T^k)\text{-chains } \zeta^k \text{ in } \mathcal{E} \text{ such that the chain} \\ \text{exponents satisfy } \lambda(\zeta^k) \to \lambda \text{ as } k \to \infty \end{array} \right\}$$

  Here $\mathcal{E}$ denotes the *lift* of the chain control set E.

Clearly we have $\Sigma_{Fl} \subset \Sigma_{Ly}$. The next Theorem shows, that the Lyapunov spectrum is *sandwiched* between the Floquet and the Morse spectrum, i.e. we have

$$\Sigma_{Fl} \subset \Sigma_{Ly} \subset \Sigma_{Mo},$$

and it characterizes the various spectral terms more closely.

Theorem 1.2.9 *Consider the bilinear system (1.10) under the Lie algebra rank condition (1.12).*

(a) *For each main control set D of the projected control system (1.11)* $\operatorname{cl}\Sigma_{Fl}(D)$ *is a bounded interval. Each* $\lambda \in \Sigma_{Fl}(D)$ *is of the form* $\lambda \in \operatorname{spec}(g)$ *for some* $g \in \operatorname{int}\mathcal{S}$.

(b) *For each chain control set E of the projected system (1.11)* $\Sigma_{Mo}(E) = [\kappa^*(E), \kappa(E)]$ *is a closed, bounded interval. The order* $E_i \prec E_j$ *implies* $\kappa^*(E_i) < \kappa(E_j)$ *and* $\kappa(E_i) < \kappa(E_j)$.

(c) $\Sigma_{Ly} \subset \Sigma_{Mo}$ *and each* $\kappa^*(E_j), \kappa(E_j)$ *for* $j = 1, \ldots, l$ *are regular Lyapunov exponents* $\lambda(u, x)$ *for some* $(u, x) \in \mathcal{U} \times \mathbb{R}^d$ *with* $\mathbb{P}\eta(t, 0, u)\mathbb{P}x \in E_j$ *for all* $t \geq 0$.

(d) *If the projected control system (1.11) is completely controllable on* $\mathbb{P}^{d-1}$, *then*

$$\operatorname{cl}\Sigma_{Fl}(\mathbb{P}^{d-1}) = \Sigma_{Ly} = \Sigma_{Mo}(\mathbb{P}^{d-1}).$$

Proof. Cf. Theorem 7.3.22 and Corollaries 7.3.18 and 7.3.23 in [9]. ■

Next we analyze, when the closure of the Floquet spectrum, the Lyapunov spectrum and the Morse spectrum coincide. Consider the family of bilinear systems

$$\begin{array}{l} \dot{x} = f_0(x) + \sum_{i=1}^m u_i(t) f_i(x) \\ u \in \mathcal{U}^\rho = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U^\rho \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\} \\ U^\rho = \rho \cdot U, \rho \geq 0, \end{array} \tag{1.15}$$

where $U \subset \mathbb{R}^m$ is convex, compact with $0 \in \operatorname{int} U$. We also use $\rho = \infty$, thus $\mathcal{U}^\infty = \bigcup_{\rho>0} \mathcal{U}^\rho = L_\infty(\mathbb{R}, \mathbb{R}^m)$. For each $\rho \in [0, \infty]$ the objects related to $(1.15)^\rho$ may be denoted by a subscript ρ. Then we define the ρ-inner pair condition as

Condition 1.2.10 *For all* $\rho, \rho' \in [0, \infty]$ *with* $\rho < \rho'$ *and all chain control sets* E^ρ *of* $(1.15)^\rho$ *each point* $(u, x) \in \mathcal{E}^\rho$ *is an inner pair of* $(1.15)^{\rho'}$.

We get the following result.

Theorem 1.2.11 *Consider the family* $(1.15)^\rho$ *of bilinear control systems , and assume that for each* ρ *the Lie algebra rank condition (1.12) and the* ρ*-inner pair condition 1.2.10 is fulfilled. Then there are at most* $d-1$ *values for* $\rho \in [0, \infty]$ *such that for all other* $\rho \in [0, \infty]$ *we have* $k(\rho) = l(\rho)$, *where* $k(\rho)$ *denotes the number of the main control sets and* $l(\rho)$ *the number of the chain control sets. Furthermore*

$$\operatorname{cl} D_i^\rho = E_i^\rho \quad \textit{and} \quad \operatorname{cl}\Sigma_{Fl}(D_i^\rho) = \Sigma_{Ly}(D_i^\rho) = \Sigma_{Mo}(E_i^\rho) \textit{ for } i = 1, \ldots, k(\rho).$$

Proof. Follows from Lemma 6.3.3 and Theorem 6.3.4 in [9] together with Theorem 7.1.5 in [9]. ■

If the state space of our bilinear control system is two dimensional, then we get a result on the spectra without the ρ-inner pair condition. This will be used in Chapter 5.1 to compute analytically the spectrum of the perturbed Duffing-van der Pol oscillator.

Theorem 1.2.12 *Consider the bilinear system (1.10) on* $\mathbb{R}^2$ *and suppose, that the Lie algebra rank condition (1.12) is fulfilled. Then its spectrum satisfies* $\mathrm{cl}\,\Sigma_{Fl} = \Sigma_{Ly} = \Sigma_{Mo}$. *More precisely, we have the following three possibilities:*

(a) *The projected system (1.11) has two main control sets* $D_1 \prec D_2$ *and two chain control set* $E_1 = \mathrm{cl}\,D_1, E_2 = \mathrm{cl}\,D_2$. *Then*

$$\mathrm{cl}\,\Sigma_{Fl} = \bigcup_{i=1}^{2} \mathrm{cl}\,\Sigma_{Fl}(D_i) = \Sigma_{Ly} = \bigcup_{j=1}^{2} \Sigma_{Mo}(E_j)$$

and the entire spectrum consists of real eigenvalues of constant matrices.

(b) *The projected system (1.11) has two main control sets* $D_1 \prec D_2$ *and one chain control set* $E = \mathbb{P}^{d-1}$. *Then*

$$\mathrm{cl}\,\Sigma_{Fl} = \bigcup_{i=1}^{2} \mathrm{cl}\,\Sigma_{Fl}(D_i) = \Sigma_{Ly} = \Sigma_{Mo}(E)$$

and the entire spectrum consists of real eigenvalues of constant matrices.

(c) *The projected system (1.11) has one main control sets* D *and hence one chain control set* $E = \mathbb{P}^{d-1}$. *Then*

$$\mathrm{cl}\,\Sigma_{Fl}(D) = \Sigma_{Ly} = \Sigma_{Mo}(E)$$

and the spectrum of constant matrices may be contained strictly in Σ_{Ly}.

Proof. See Theorem 10.1.1. in [9]. ■

Finally the next Theorem shows, when it is actually enough to compute the eigenvalues of *constant matrices* to compute the whole Lyapunov spectrum.

Theorem 1.2.13 *Consider the bilinear system (1.10) and its projected system (1.11) satisfying the Lie algebra rank condition (1.12). Let* E *be a chain control set, with multiplicity* 1. *Then*

$$\Sigma_{Mo}(E) = \{\lambda \in \Sigma_{Mo}(E) : \lambda \text{ is an eigenvalue of } A(u) \text{ for some } u \in U\},$$

and the main control set D *with* $\mathrm{cl}\,D = E$ *satisfies*

$$\mathrm{cl}\,\Sigma_{Fl}(D) = \Sigma_{Ly}(D) = \Sigma_{Fl}(E).$$

Proof. See Theorem 7.3.25. in [9]. ■

The following technical proposition will be used in Section 4.3. As we have seen in Theorem 1.2.4, the eigenspaces of elements in $\mathrm{int}\,\mathcal{S}_{\leq t}$ lie in the interior of main control sets. If one takes the direct sum of all those eigenspaces, which do not lie in $C := D_k$, then this sum does not intersect C.

Proposition 1.2.14 *Consider the bilinear control system (1.10) and its projected system (1.11) satisfying the Lie algebra rank condition (1.12). Let* $E_1, \dots, E_l$ *be the chain control sets of the projected system (1.11) with* $l \geq 2$ *and suppose, that* E_l *has multiplicity* 1*. Let* $D_1 \prec \dots \prec D_{k-1} \prec D_k =: C$ *be the main control sets with* $C \subset E_l$*. Suppose, that*

$$\begin{array}{rcll} \Sigma_{Ly}(D_i) & = & [\kappa_i^*, \kappa_i] & \text{for } i = 1, \dots, k-1 \text{ and} \\ \Sigma_{Ly}(C) & = & [\kappa_k^*, \kappa_k] & \text{with } \kappa_k^* > \kappa_i \text{ for } i = 1, \dots, k-1. \end{array}$$

Let $g \in \operatorname{int} \mathcal{S}_{\leq t}$ *for some* $t > 0$ *and* $\operatorname{spec}(g) = \{\sigma_1, \dots, \sigma_n\}$ *with*

$$\operatorname{Re}(\ln \sigma_1) \leq \dots \leq \operatorname{Re}(\ln \sigma_{n-1}) < \operatorname{Re}(\ln \sigma_n).$$

Then

$$C \cap \mathbb{P}(\oplus_{i=1}^{n-1} E(\sigma_i)) = \emptyset.$$

Proof. Suppose, that there is an $x \in \oplus_{i=1}^{n-1} E(\sigma_i), x \neq 0$ such that $\mathbb{P}x \in C$.

Then there is an $u = u_g \in \mathcal{U}$ piecewise constant and periodic with period Θ, such that $\eta(\Theta, 0, u) = g$. Now we use Floquet Theory, as in G.Sansone and R.Conti [27]. There is a matrix $Q \in \mathcal{L}(\mathbb{C})$ with $\eta(\Theta, 0, u) = e^{\Theta Q}$. Note that because $\eta(\Theta, 0, u) = \overline{\eta(\Theta, 0, u)}$ if follows, that $g^2 = e^{\Theta Q}\overline{e^{\Theta Q}} = e^{\Theta(Q+\bar{Q})}$. Thus if we define $R := Q + \bar{Q} \in \mathbb{R}^{d \times d}$ we have $g^2 = e^{\Theta R}$ and

$$\eta(t, 0, u) = G(t) e^{\frac{1}{2} t R}$$

by defining $G(t) := \eta(t, 0, u) e^{-\frac{1}{2} t R} \in gl(\mathbb{R}^d)$. $G(t)$ is 2Θ-periodic. Denote the eigenvalues of Q by $\operatorname{spec}(Q) = \{\xi_1^Q, \dots, \xi_n^Q\}$ and the eigenvalues of R by $\operatorname{spec}(R) = \{\xi_1, \dots, \xi_n\}$. If we order them in the right way we get

$$\sigma_i = e^{\Theta \xi_i^Q}.$$

But because σ_i^2 are the eigenvalues of g^2 we have $\sigma_i^2 = e^{\Theta \xi_i}$ and get

$$\sigma_i = e^{\frac{1}{2}\Theta \xi_i} \text{ for } i = 1, \dots, n.$$

In particular we have $\operatorname{Re}\xi_1 \leq \dots \leq \operatorname{Re}\xi_{n-1} < \operatorname{Re}\xi_n$ and $E(\sigma_i) = E(\xi_i)$.

Now let $\{x_{ij}\}_{j=1,\dots l_i} \subset \mathbb{R}^d$ be a basis of $E(\xi_i), i = 1, \dots, n$ and let

$$x = \sum_{i=1}^{n-1} \sum_{j=1}^{l_j} a_{ij} x_{ij}$$

with $a_{ij} \in \mathbb{R}$, where $\dim E(\xi_i) = l_i$. Let there be without loss of generality a $j \in \{1, \dots, l_{n-1}\}$ such that $a_{n-1,j} \neq 0$. Let $i_0 \in \{1, \dots, n-1\}$ be the smallest number such that $\operatorname{Re}(\xi_{i_0}) = \operatorname{Re}(\xi_{n-1})$.

If $i_0 = 1$, then $\operatorname{Re}(\xi_1) = \dots = \operatorname{Re}(\xi_{n-1})$. Now because $g \in \operatorname{int} \mathcal{S}_{\leq t}$, it follows, that $\mathbb{P}(\oplus_{i=1}^{n-1} E(\xi_i)) \subset \operatorname{int} D$ for some main control set D with $D \prec C$, see Proposition 7.3.7 in

[9]. Thus it would follow, that $\mathbb{P}x \in C \cap \operatorname{int} D$ which is a contradiction to the maximality of the control sets.

Now suppose, that $i_0 > 1$. If there is no $i \in \{1, \dots, i_0 - 1\}$ such that there is an $j \in \{1, \dots, l_i\}$ with $a_{ij} \neq 0$, then we get as before, that $\mathbb{P}x \in \operatorname{int} D$ for a control set $D \prec C$, and we get a contradiction again.

Now suppose, that there is an $i \in \{1, \dots, i_0 - 1\}$ such that there is an $j \in \{1, \dots, l_i\}$ with $a_{ij} \neq 0$. Consider the linear autonomous differential equation on $\mathbb{R}^d$

$$\dot{y} = Ry,$$

where we denote the fundamental solution by $\eta(t, \tau, R) \in gl(\mathbb{R}^d), t, \tau \in \mathbb{R}$. Define $x_1 := \sum_{i=1}^{i_0-1} \sum_{j=1}^{l_i} a_{ij} x_{ij}$ and $x_2 := \sum_{i=i_0}^{n-1} \sum_{j=1}^{l_i} a_{ij} x_{ij}$ and

$$x_1(t) := \eta(t, 0, R)x_1 \text{ and } x_2(t) := \eta(t, 0, R)x_2.$$

Note, that $x_1(t) \in \oplus_{i=1}^{i_0-1} E(\xi_i)$ and $x_2(t) \in \oplus_{i=i_0}^{n-1} E(\xi_i)$ for all $t \in \mathbb{R}$. We finally denote

$$x(t) := \eta(t, 0, R)x = x_1(t) + x_2(t).$$

For every $\gamma \in (\operatorname{Re}(\xi_{i_0-1}), \operatorname{Re}(\xi_{i_0}))$ there are $K_1, K_2 > 0$ such that

$$\begin{aligned} \|x_1(t)\| &< K_1 e^{\gamma t} \quad \text{and} \\ \|x_2(t)\| &> K_2 e^{\gamma t} \quad \text{for all } t \geq 0. \end{aligned} \tag{1.16}$$

A metric d on $\mathbb{P}^{d-1}$ is given by

$$d(\mathbb{P}x, \mathbb{P}y) := \min \left\{ \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|, \left\| -\frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \right\}$$

for all $x, y \in \mathbb{R}^d \setminus 0$, which makes $\mathbb{P}^{d-1}$ to a compact metric space. We show, that for every $\varepsilon > 0$ there is a $N > 0$ such that for all $j > N$ we have

$$\inf\{d(\mathbb{P}\eta(j\Theta, 0, u)x, \mathbb{P}y) : \mathbb{P}y \in \mathbb{P}(\oplus_{i=i_0}^{n-1} E(\xi_i))\} < \varepsilon.$$

We have

$$\left\| \frac{x(t)}{\|x(t)\|} - \frac{x_2(t)}{\|x_2(t)\|} \right\| \leq \|x(t)\| \left| \frac{1}{\|x(t)\|} - \frac{1}{\|x_2(t)\|} \right| + \frac{\|x_1(t)\|}{\|x_2(t)\|}.$$

Because $\|x_2(t)\| - \|x_1(t)\| > 0$ for t big enough we get by (1.16)

$$\frac{\|x_1(t)\|}{\|x(t)\|} = \frac{\|x_1(t)\|}{\|x_1(t) + x_2(t)\|} \leq \frac{\|x_1(t)\|}{\|x_2(t)\| - \|x_1(t)\|}$$

and it follows again with (1.16), that $\lim_{t \to \infty} \frac{\|x_1(t)\|}{\|x(t)\|} = 0$.

Furthermore we have

$$\|x(t)\| \left| \frac{1}{\|x(t)\|} - \frac{1}{\|x_2(t)\|} \right| = \|x(t)\| \left| \frac{\|x_2(t)\| - \|x(t)\|}{\|x(t)\| \, \|x_2(t)\|} \right| = \frac{|\|x_2(t)\| - \|x(t)\||}{\|x_2(t)\|}$$

If $\|x_2(t)\| - \|x(t)\| \geq 0$ we get for t big enough

$$\frac{\|x_2(t)\| - \|x(t)\|}{\|x_2(t)\|} \leq \frac{\|x_2(t)\| - |\|x_2(t)\| + \|x_1(t)\||}{\|x_2(t)\|} = \frac{\|x_1(t)\|}{\|x_2(t)\|}$$

and if $\|x_2(t)\| - \|x(t)\| < 0$ we get for t big enough

$$\frac{-\|x_2(t)\| + \|x(t)\|}{\|x_2(t)\|} \leq \frac{-\|x_2(t)\| + \|x_1(t)\| + \|x_2(t)\|}{\|x_2(t)\|} = \frac{\|x_1(t)\|}{\|x_2(t)\|}.$$

Because of (1.16) it follows $\lim_{t\to\infty} \frac{\|x_1(t)\|}{\|x_2(t)\|} = 0$ and we get $\lim_{t\to\infty} \|x(t)\| \left| \frac{1}{\|x(t)\|} - \frac{1}{\|x_2(t)\|} \right| = 0$. We finally obtain

$$\lim_{t\to\infty} \left\| \frac{x(t)}{\|x(t)\|} - \frac{x_2(t)}{\|x_2(t)\|} \right\| = 0. \tag{1.17}$$

Because $\eta(t,0,u) = G(t)\eta(t,0,R)$ we have $\eta(t,0,u)x = G(t)x(t)$ and $\eta(t,0,u)x_2 = G(t)x_2(t)$. By periodicity of $G(t)$ and with (1.17) we get

$$\lim_{i\to\infty} \left\| \frac{\eta(i\Theta,0,u)x}{\|\eta(i\Theta,0,u)x\|} - \frac{\eta(i\Theta,0,u)x_2}{\|\eta(i\Theta,0,u)x_2\|} \right\| = 0.$$

Thus for every $\varepsilon > 0$ there is an $N > 0$ such that for all $i > N$ we have

$$\inf\{d(\mathbb{P}\eta(i\Theta,0,u)x, \mathbb{P}y) : \mathbb{P}y \in \mathbb{P}(\oplus_{i=i_0}^{n-1} E(\xi_i))\} < \varepsilon. \tag{1.18}$$

Now according to Proposition 7.3.7 in [9] we have $\mathbb{P}(\oplus_{i=i_0}^{n-1} E(\xi_i)) \subset \operatorname{int} D$ for some main control set $D \prec C$. But relation (1.18) means, that there is a $t > 0$ such that $\mathbb{P}\eta(t,0,u)x \in \operatorname{int} D$, thus it would follow, that $C \prec D$ which is a contradiction. ∎

Chapter 2

The Nonexistence of Control Sets

In this chapter we start to analyze the controllability properties of a control affine system at a singular point x^*. Here (and in the following chapters) the Lyapunov spectrum serves as an indicator for the controllability properties.

We first consider systems, where the supremum of the Lyapunov spectrum is negative, or the infimum of the Lyapunov spectrum is positive. Both cases are treated together in the first section of this chapter. We denote the class of systems where the supremum of the Lyapunov spectrum is negative as *stable* and the second class where the infimum of the Lyapunov spectrum is positive as *unstable* systems. We show in Theorem 2.1.4, that in both cases there is an open neighborhood around the singular point x^*, such that no control set with nonvoid interior intersects this neighborhood. For showing this, we need the notion of local stable and unstable fibre bundles for the control flow of the nonlinear control system. In a more general setting (for linear flows on vector bundles) this can be found in I.U.Bronstein and A.Y.Kopanskii [7] and in Colonius and Kliemann [9], Chapter 5. The necessary results we need here were taken from Chapter 7 in [9]. Local stable fibre bundles for the control flow are subbundles of $\mathcal{U} \times \mathbb{R}^d$. They can be characterized locally by the long time behavior of trajectories starting near the singular point.

In the second part we consider the case, where we have two exponentially separated subbundles $\mathcal{V}^+$ and $\mathcal{V}^-$ of $\mathcal{U} \times \mathbb{R}^d$ with $\mathcal{U} \times \mathbb{R}^d = \mathcal{V}^+ \oplus \mathcal{V}^-$ which are invariant with respect to the linearized control flow and $\sup \Sigma_{Ly}(\mathcal{V}^+) < 0 < \inf \Sigma_{Ly}(\mathcal{V}^-)$. Here the dichotomy spectrum of the linearized flow which will be defined below, plays an important part. This dichotomy spectrum of the linearized flow will be compared with the dichotomy term of the Appendix 6.1.1 for linear time variant differential equations. If we consider the linearization of the nonlinear control system at the singular point for a fixed control function, then we obtain a linear time variant differential equation. There the dichotomy spectrum is given by Definition 6.1.3. We show in Lemma 2.2.4, that this dichotomy spectrum (for an individual control function) is contained in the dichotomy spectrum of the linearized flow.

Then we can show in Theorem 2.2.6, that there is a neighborhood around the singular point with the following property: Given two points of a control set which lie in this neighborhood, then one can steer if at all one point to the other only by leaving this neighborhood at some time.

2.1 Stable and Unstable Systems

In this section we consider control affine systems with a singular point, such that the corresponding Lyapunov spectrum has the property, that either $\sup \Sigma_{Ly} < 0$ or $\inf \Sigma_{Ly} > 0$. We will show, that locally around the singular point there is no control set with nonvoid interior. The idea behind the proof is to show, that if we have for example $\sup \Sigma_{Ly} < 0$, then all trajectories $\varphi(t, 0, x, u)$ which start near the singular point tend in an uniform way to the singular point as t goes to infinity. Thus there can not be periodic trajectories near the singular point, and therefore no control sets with nonvoid interior. For getting the uniformity result we need the notion of stable and unstable fibre bundles.

We start by introducing the necessary notations. We consider a control-affine system on $\mathbb{R}^d$ with the following form:

$$\begin{array}{l} \dot{x} = f_0(x) + \sum_{i=1}^{m} u_i(t) f_i(x) \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\} \end{array} \tag{2.1}$$

where U is a compact and convex subset of $\mathbb{R}^m$ containing 0. We assume that $f_0, \dots, f_m$ are C^2 vector fields on $\mathbb{R}^d$. Furthermore, suppose that for all $(u, x) \in \mathcal{U} \times \mathbb{R}^d$ the equation (2.1) has a unique solution $\varphi(t, \tau, x, u), t, \tau \in \mathbb{R}$, with $\varphi(\tau, \tau, x, u) = x$.

We suppose in this section, that the system (2.1) has a singular point $x^* \in \mathbb{R}^d$.

The system linearized at the singular point $x^* \in \mathbb{R}^d$ has the form

$$\begin{array}{l} \dot{x} = A_0 x + \sum_{i=1}^{m} u_i(t) A_i x \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\}, \end{array} \tag{2.2}$$

where $A_i := \left.\frac{\partial f_i}{\partial x}\right|_{x=x^*}$. We denote the fundamental solution of (2.2) for a $u \in \mathcal{U}$ by $\eta(t, \tau, u)$, with $\eta(\tau, \tau, u)x = x$, where $\tau, t \in \mathbb{R}, x \in \mathbb{R}^d$. Note, that this defines a mapping $\mathbb{R} \times \mathbb{R} \times \mathcal{U} \to gl(\mathbb{R}^d)$, $(t, \tau) \longmapsto \eta(t, \tau, u)$.

Associated with the bilinear control system is its projection on the projective space $\mathbb{P}^{d-1}$,

$$\begin{array}{l} \dot{p} = h(p, u(t)) = h_0(p, u(t)) + \sum_{i=1}^{m} u_i(t) h_i(p, u(t)) \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a } t \in \mathbb{R}, \text{locally integrable}\} \end{array} \tag{2.3}$$

where

$$h_i(p) := (A_i - p^T A_i p \cdot \text{id})p \text{ for all } i = 0, \dots, m.$$

We denote the solution of (2.3) by $\mathbb{P}\eta(t, \tau, u)p$, with $\mathbb{P}\eta(\tau, \tau, u)p = p$, where $\tau, t \in \mathbb{R}, p \in \mathbb{P}^{d-1}, u \in \mathcal{U}$.

We remind you, that the accessibility rank condition for the projected system (2.3) is given by

$$\dim \mathcal{LA}\{h(\cdot,u) : u \in U\}(p) = d-1 \text{ for all } p \in \mathbb{P}^{d-1}, \tag{2.4}$$

cf. Chapter 1.2.1.

The control flow associated with (2.1) is denoted by

$$\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d \to \mathcal{U} \times \mathbb{R}^d.$$

and the control flow (or the linearized flow) associated with the linearized system (2.2) is

$$\begin{array}{rccc} \mathbf{T}\Phi: & \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d & \to & \mathcal{U} \times \mathbb{R}^d \\ & (t,u,x) & \mapsto & (u(t+\cdot), \eta(t,0,u)x). \end{array} \tag{2.5}$$

In this section we will get results on control affine system with $\sup \Sigma_{Ly} < 0$ or $\inf \Sigma_{Ly} > 0$. But before looking at these systems more closely, we make a short trip to stable and unstable fibre bundles for the nonlinear control flow. Here we first suppose, that we have a decomposition of $\mathcal{U} \times \mathbb{R}^d$ into two exponentially separated subbundles $\mathcal{V}^+$ and $\mathcal{V}^-$ with $\mathcal{U} \times \mathbb{R}^d = \mathcal{V}^+ \oplus \mathcal{V}^-$. By Theorem 1.2.7 such a decomposition is possible, if the projected system (2.3) has at least two chain control sets.

Then the following local stable manifold theorem for the system (2.1) holds.

Theorem 2.1.1 *Consider the control system (2.1) with singular point $x^* \in \mathbb{R}^d$. Assume that the linearized flows $\mathbf{T}\Phi$ on $\mathcal{U} \times \mathbb{R}^d$ admits the decomposition*

$$\mathcal{U} \times \mathbb{R} = \mathcal{V}^+ \oplus \mathcal{V}^- \tag{2.6}$$

such that there are constants $c_0 > 0$ and $\varepsilon_0 > 0$ with

$$|\mathbf{T}\Phi_t(u,x)| \le c_0 \exp(-\varepsilon_0 t) \, |\mathbf{T}\Phi_t(u,y)|, \qquad t \ge 0 \tag{2.7}$$

for $(u,x) \in \mathcal{V}^+, (u,y) \in \mathcal{V}^-$ with $\|x\| = \|y\| = 1$, and

$$\kappa^+ := \sup \Sigma_{Ly}(\mathcal{V}^+) < 0. \tag{2.8}$$

We call $\mathcal{V}^+$ satisfying (2.8) the stable subbundle *and $\mathcal{V}^-$ the* unstable subbundle . *Then there are $\delta > 0$ and a map*

$$S^+ : \big\{(u,x) \in \mathcal{V}^+ : \|x\| < \delta\big\} \to \mathcal{U} \times \mathbb{R}^d$$

of the form

$$S^+(u,x) = (u, s^+(u,x))$$

with the following properties:

(a) *For every* $\alpha > \kappa^+$ *and every* (u, y) *in*

$$\mathcal{W}^+ := \left\{S^+(u,x) : (u,x) \in \mathcal{V}^+ \text{ and } \|x\| < \delta\right\} = \operatorname{im} S^+$$

one has

$$\lim_{t\to\infty} e^{-\alpha t}(\varphi(t,0,x,u) - x^*) = 0.$$

We call $\mathcal{W}^+$ *a* local stable manifold corresponding to the stable subbundle $\mathcal{V}^+$.

(b) *The map* S^+ *is a bundle isomorphism onto its image* $\mathcal{W}^+$*; in particular, for every control* $u \in \mathcal{U}$ *the fibres*

$$W_u^+ := \left\{x \in \mathbb{R}^d : (u,x) \in \mathcal{W}^+\right\}$$

are topological manifolds and their dimension equals the dimension of $\mathcal{V}^+$.

(c) *The local stable manifold* $\mathcal{W}^+$ *is positively invariant under the control flow* Φ*, i.e. for* $(u,x) \in \mathcal{W}^+$ *one has*

$$(\theta_t u, \varphi(t,0,x,u)) \in \mathcal{W}^+ \text{ for all } t \geq 0.$$

(d) *The distance of the subbundle* $\mathcal{W}^+$ *to* $\mathcal{V}^+$ *can be made arbitrary small in the following sense by choosing* $\delta > 0$ *small: For all* $h > 0$ *there is a* $\delta > 0$ *such that* $\mathcal{W}^+$ *is contained in the cone* $K_h(\mathcal{V}^+)$ *of angle* h *around* $\mathcal{V}^+$ *given by*

$$K_h(\mathcal{V}^+) := \left\{(u, x^+ + x^-) : \begin{array}{c}(u,x^+) \in \mathcal{V}^+ \text{ and } (u,x^-) \in \mathcal{V}^- \\ \text{with } \|x^-\| \leq h\,\|x^+\|\end{array}\right\}.$$

Proof. See Theorem 7.4.1 in [9]. ■

Remark 2.1.2 *Under the same assumptions as in Theorem 2.1.1, by replacing (2.8) with*

$$\kappa^- := \inf \Sigma_{Ly}(\mathcal{V}^-) > 0,$$

one can prove an unstable manifold theorem via a map

$$S^- : \left\{(u,x) \in \mathcal{V}^- : |x| < \delta\right\} \to \mathcal{U} \times \mathbb{R}^d$$

and its image $\operatorname{im} S^- =: \mathcal{W}^-$*. Then the local unstable manifold* $\mathcal{W}^-$ *is negatively invariant, the fibres are topological manifolds and for every* $(u,x) \in \mathcal{W}^-$ *and every* $\alpha < \kappa^-$ *we have*

$$\lim_{t\to-\infty} e^{-\alpha t}(\varphi(t,0,x,u) - x^*) = 0.$$

If one of the bundles $\mathcal{V}^+$ or $\mathcal{V}^-$ is trivial, we get the following result.

Corollary 2.1.3 *If*

$$\sup \Sigma_{Ly} < 0 \tag{2.9}$$

then there exists a $\sigma > 0$ *such that for all* $x \in B_\sigma(x^*)$ *and all* $u \in \mathcal{U}$ *we have* $x \in \mathcal{W}_u^+$.

If

$$\inf \Sigma_{Ly} > 0 \tag{2.10}$$

then there exists a $\sigma > 0$ *such that for all* $x \in B_\sigma(x^*)$ *and all* $u \in \mathcal{U}$ *we have* $x \in \mathcal{W}_u^-$.

Proof. This follows by Theorem 2.1.1 (b) and Remark 2.1.2 by considering in the first case the subbundles $(\mathcal{V}^+, 0) = (\mathcal{U} \times \mathbb{R}^d, 0)$ and $(\mathcal{V}^-, 0) = (\mathcal{U} \times \mathbb{R}^d, 0)$ in the second case. ■

By Corollary 7.2.17 in [9] follows, that if the projected control system (2.3) fulfills the Lie algebra rank condition (2.4), it follows, that

$$\sup \Sigma_{Ly} = \sup \Sigma_{Fl} \text{ and } \inf \Sigma_{Ly} = \inf \Sigma_{Fl}.$$

Thus in this case we can replace the relation (2.9) by $\sup \Sigma_{Fl} < 0$ and the relation (2.10) by $\inf \Sigma_{Fl} < 0$.

Corollary 2.1.3 means, that if we have $\sup \Sigma_{Ly} < 0$, then we get

$$\lim_{t\to\infty} \|\varphi(t, 0, x, u) - x^*\| = 0$$

and if we have $\inf \Sigma_{Ly} > 0$, then we get

$$\lim_{t\to-\infty} \|\varphi(t, 0, x, u) - x^*\| = 0$$

for all $x \in B_\sigma(x^*)$ and all $u \in \mathcal{U}$. Hence if $\sup \Sigma_{Ly} < 0$, then the singular point x^* is locally attractive for all $u \in \mathcal{U}$, and a subset of its domain of attraction is given by $B_\sigma(x^*)$. The important fact is, that this subset of the domain of attraction is independent of the chosen $u \in \mathcal{U}$.

The next theorem shows, that locally around the singular point there is no control set with nonvoid interior, if the Lyapunov spectrum of the bilinear control system (2.2) fulfills the relation (2.9) or (2.10).

Theorem 2.1.4 *Consider the nonlinear control system*

$$\begin{array}{l} \dot{x} = f_0(x) + \sum_{i=1}^m u_i(t) f_i(x) \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\} \end{array} \tag{2.11}$$

where U *is a compact and convex subset of* $\mathbb{R}^m$ *containing* 0 *and* $f_0, \dots, f_m$ *are* C^2 *vector fields on* $\mathbb{R}^d$ *and its associated linearized system (linearized at the singular point* $x^* \in \mathbb{R}^d$*)*

$$\begin{array}{l} \dot{x} = A_0 x + \sum_{i=1}^m u_i(t) A_i x \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\} \end{array} \tag{2.12}$$

where $A_i := \left.\frac{\partial f_i}{\partial x}\right|_{x=x^*}$. *Suppose, that the following conditions are satisfied.*

(a) *The nonlinear control system (2.11) has a singular point* $x^* \in \mathbb{R}^d$ *and it is locally accessible from all* $x \in \mathbb{R}^d \setminus \{x^*\}$.

(b) *The Lyapunov spectrum of the bilinear control system (2.12) fulfills*

$$\sup \Sigma_{Ly} < 0 \text{ or } \inf \Sigma_{Ly} > 0.$$

Then there exists a neighborhood N *of* x^* *such that there is no control set* $D \subset \mathbb{R}^d$ *with nonvoid interior and* $N \cap \operatorname{int} D \neq \emptyset$.

Proof. First suppose, that $\sup \Sigma_{Fl} < 0$. Then according to Corollary 2.1.3 there is a $\sigma > 0$ such that for all $u \in \mathcal{U}$ and all $x \in B_\sigma(x^*)$ we have $x \in \mathcal{W}_u^+$. The idea is to show, that there can not be a periodic trajectory, which lives in $B_\sigma(x^*)$. Now suppose, that there is a control set $D \subset \mathbb{R}^d$ with nonvoid interior, such that $B_\sigma(x^*) \cap \operatorname{int} D \neq \emptyset$, i.e. there are $x, y \in B_\sigma(x^*) \cap \operatorname{int} D$. Because the system is locally accessible from all $x \in \mathbb{R}^d \setminus \{0\}$ the interior of D is controllable. Hence for $x, y \in \operatorname{int} D$ there are $t_i > 0, u_i \in \mathcal{U}, i = 1, 2$ such that

$$\begin{aligned} \varphi(t_1, 0, x, u_1) &= y, \\ \varphi(t_2, 0, y, u_2) &= x. \end{aligned}$$

Define the function $u \in \mathcal{U}$ by

$$u(t) := \begin{cases} u_1(t) & \text{for } t \in [0, t_1), \\ u_2(t - t_1) & \text{for } t \in [t_1, t_1 + t_2), \end{cases}$$

on $[0, t_1 + t_2)$ and continue it periodically. By construction the trajectory $\varphi(t, 0, x, u)$ is $(t_1 + t_2)$-periodic. Because $x \in B_\sigma(x^*)$ it follows that $x \in \mathcal{W}_u^+$. But this is a contradiction, because according to Theorem 2.1.1 we must have $\lim_{t \to \infty} \|\varphi(t, 0, x, u) - x^*\| = 0$.

Next suppose, that $\inf \Sigma_{Fl} > 0$. According to Corollary 2.1.3 there is a $\sigma > 0$ such that for all $u \in \mathcal{U}$ and all $x \in B_\sigma(x^*)$ we have $x \in \mathcal{W}_u^-$. Suppose, that there is a control set $D \subset \mathbb{R}^d$ with nonvoid interior, such that $B_\sigma(x^*) \cap \operatorname{int} D \neq \emptyset$, i.e. there are $x, y \in B_\sigma(x^*) \cap \operatorname{int} D$. Since we have local accessibility on $\mathbb{R}^d \setminus \{0\}$, we get $D = \operatorname{cl} \mathcal{O}^+(x) \cap \mathcal{O}^-(x) = \operatorname{cl} \mathcal{O}^+(y) \cap \mathcal{O}^-(y)$ (cf. Lemma 3.2.13 in [9]). Hence there are $t_i > 0, u_i \in \mathcal{U}, i = 1, 2$ such that

$$\begin{aligned} y &= \varphi(-t_1, 0, x, u_1) \\ x &= \varphi(-t_2, 0, y, u_2). \end{aligned}$$

Define the function $u \in \mathcal{U}$ by

$$u(t) := \begin{cases} u_1(t) & \text{for } t \in (-t_1, 0], \\ u_2(t + t_1) & \text{for } t \in (-t_2 - t_1, -t_1], \end{cases}$$

on $(-t_1 - t_2, 0]$ and continue it periodically. Hence the trajectory $\varphi(t, 0, x, u)$ is $(t_1 + t_2)$-periodic. Now because $x \in B_\sigma(x^*)$ it follows that $x \in \mathcal{W}_u^-$. This is a contradiction, because according to Theorem 2.1.1 and Remark 2.1.2 we have $\lim_{t \to -\infty} \|\varphi(t, 0, x, u) - x^*\| = 0$. ∎

2.2 Hyperbolic Systems

We consider the control-affine system on $\mathbb{R}^d$

$$\begin{array}{l} \dot{x} = f_0(x) + \sum_{i=1}^{m} u_i(t) f_i(x) \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\}, \end{array} \tag{2.13}$$

where U is a compact and convex subset of $\mathbb{R}^m$ containing 0. We assume that $f_0, \dots, f_m$ are C^2 vector fields on $\mathbb{R}^d$. Furthermore, suppose that for all $(u,x) \in \mathcal{U} \times \mathbb{R}^d$ the system (1.1) has a unique solution $\varphi(t,\tau,x,u), t,\tau \in \mathbb{R}$, with $\varphi(\tau,\tau,x,u) = x$.

We suppose, that the system (2.1) has the singular point $x^* = 0 \in \mathbb{R}^d$. Note that the assumption $x^* = 0$ is no restriction. If our control affine system has a singular point $x^* \neq 0$ we can transform the control system by an affine transformation into a control affine system with 0 as singular point. The assumption $x^* = 0$ is made here only for notational convenience.

The system linearized at the singular point $x^* \in \mathbb{R}^d$ has the form

$$\begin{array}{l} \dot{x} = A_0 x + \sum_{i=1}^{m} u_i(t) A_i x \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\} \end{array} \tag{2.14}$$

where $A_i := \left.\frac{\partial f_i}{\partial x}\right|_{x=x^*}$. We denote the fundamental solution of (2.2) for a $u \in \mathcal{U}$ by $\eta(t,\tau,u)$, with $\eta(\tau,\tau,u)x = x$, where $\tau, t \in \mathbb{R}, x \in \mathbb{R}^d$.

The control flow associated with (2.13) is denoted by

$$\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d \to \mathcal{U} \times \mathbb{R}^d,$$

and the control flow associated with the linearized system (2.14) is defined as in (2.5) by

$$\mathbf{T}\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d \to \mathcal{U} \times \mathbb{R}^d.$$

In this section, we consider the case, where we have two exponential separated subbundles $\mathcal{V}^+, \mathcal{V}^- \subset \mathcal{U} \times \mathbb{R}^d$ which are invariant with respect to the linearized flow and

$$\sup \Sigma_{Ly}(\mathcal{V}^+) < 0 < \inf \Sigma_{Ly}(\mathcal{V}^-).$$

Such an exponential splitting is possible, if the projected system on $\mathbb{P}^{d-1}$ has at least two chain control sets, cf. Theorem 1.2.7 and Theorem 1.2.8.

We now come to the exponential dichotomy for the linearized control flow $\mathbf{T}\Phi$. The dichotomy spectrum (or dynamical spectrum, how it is also called) was introduced by R.J.Sacker and G.R.Sell (cf. [25],[26]). The results for the linearized flow are taken from the book of Colonius and Kliemann [9].

A projection $\mathbf{P}$ on the vector bundle $\pi : \mathcal{U} \times \mathbb{R}^d \to \mathcal{U}$ is a continuous map $\mathbf{P} : \mathcal{U} \times \mathbb{R}^d \to \mathcal{U} \times \mathbb{R}^d$ with $\mathbf{P} \circ \mathbf{P} = \mathbf{P}$, such that the restrictions on the fibres $\mathbf{P}_u : (u, \mathbb{R}^d) \to (u, \mathbb{R}^d)$ are well defined linear maps.

Definition 2.2.1 *An exponential dichotomy of the linearized flow* $\mathbf{T}\Phi$ *on the vector bundle* $\pi : \mathcal{U} \times \mathbb{R}^d \to \mathcal{U}$ *is given by a projection* $\mathbf{P}$ *which is not the identity or zero on* $\mathcal{U} \times \mathbb{R}^d$ *such that there are constants* $K \geq 1, \alpha > 0$ *with*

$$\begin{aligned} |\mathbf{T}\Phi_t \circ \mathbf{P} \circ \mathbf{T}\Phi_{-\tau}| &\leq Ke^{-\alpha(t-\tau)} \quad \text{for } t \geq \tau, \\ |\mathbf{T}\Phi_t \circ (\mathrm{id} - \mathbf{P}) \circ \mathbf{T}\Phi_{-\tau}| &\leq Ke^{\alpha(t-\tau)} \quad \text{for } t \leq \tau. \end{aligned}$$

For each $\gamma \in \mathbb{R}, u \in \mathcal{U}$ the fundamental solution of the linear differential equation

$$\dot{x} = [A_0 + \sum_{i=1}^{m} u_i(t)A_i - \gamma\,\mathrm{id}]x. \tag{2.15}$$

will be denoted by

$$\eta^\gamma(t, \tau, u) \text{ for } t, \tau \in \mathbb{R}.$$

Note that we have for all $t \in \mathbb{R}$

$$\eta^\gamma(t, \tau, u) = e^{-\gamma(t-\tau)}\eta(t, \tau, u).$$

For $\gamma \in \mathbb{R}, t \in \mathbb{R}$ we define

$$\mathbf{T}\Phi_t^\gamma := e^{-\gamma t}\mathbf{T}\Phi_t.$$

Then we have

$$\mathbf{T}\Phi_t^\gamma(u, x) = (\theta_t u, \eta^\gamma(t, 0, u)x)$$

for all $t \in \mathbb{R}$.

Definition 2.2.2 *The dichotomy spectrum of the linearized flow* $\mathbf{T}\Phi$ *is*

$$\Sigma_{dich} = \{\gamma \in \mathbb{R} : \text{ the flow } \mathbf{T}\Phi^\gamma \text{ has no exponential dichotomy}\}.$$

The next lemma shows, that the dichotomy spectrum is just another description of the Lyapunov spectrum.

Lemma 2.2.3 *For the linearized control system (2.14) we have*

$$\Sigma_{Ly} = \Sigma_{dich}.$$

Proof. Since the flow $(\mathcal{U}, \theta)$ is chain transitive, this follows from Theorem 5.5.9 in [9]. ∎

Thus instead of using the Lyapunov spectrum, we can use the dichotomy spectrum to characterize the linearized control flow.

In Appendix 6.1.1 we introduce another dichotomy term. It is defined for linear time variant differential equations. The bilinear control system (2.14) can be interpreted as a family of linear differential equations, parametrized by the control functions $u \in \mathcal{U}$. Each of the resulting linear system has its own dichotomy spectrum. We will compare in the following lemma, how the dichotomy spectrum of each individual equation is related to the dichotomy spectrum of the linearized control flow.

Lemma 2.2.4 *Assume that for a $\gamma \in \mathbb{R}$ the linearized flow $\mathbf{T}\Phi^\gamma$ has an exponential dichotomy with a projection $\mathbf{P} : \mathcal{U} \times \mathbb{R}^d \to \mathcal{U} \times \mathbb{R}^d$ and constants $K \geq 1, \alpha > 0$ such that*

$$\begin{aligned} |\mathbf{T}\Phi_t^\gamma \circ \mathbf{P} \circ \mathbf{T}\Phi_{-\tau}^\gamma| &\leq Ke^{-\alpha(t-\tau)} \quad \text{for } t \geq \tau, \\ |\mathbf{T}\Phi_t^\gamma \circ (id - \mathbf{P}) \circ \mathbf{T}\Phi_{-\tau}^\gamma| &\leq Ke^{\alpha(t-\tau)} \quad \text{for } t \leq \tau. \end{aligned}$$

Then for each $u \in \mathcal{U}$ there is an invariant projector $R(\cdot) := R(\cdot, u) : \mathbb{R} \to \mathcal{L}(\mathbb{R}^d)$ in the sense of Definition 6.1.2 for the system (2.15) such that

$$\begin{aligned} \|\eta^\gamma(t,\tau,u)R(\tau)\| &\leq Ke^{-\alpha(t-\tau)} \quad \text{for } t \geq \tau, \\ \|\eta^\gamma(t,\tau,u)[\mathrm{id} - R(\tau)]\| &\leq Ke^{\alpha(t-\tau)} \quad \text{for } \tau \geq t. \end{aligned}$$

In particular it follows

$$\Sigma(A_0 + \sum_{i=1}^m u_i(t)A_i) \subset \Sigma_{dich}.$$

Proof. Because $\eta^\gamma(t,\tau,u) = \eta^\gamma(t-\tau, 0, \theta_\tau u)$ we have

$$\mathbf{T}\Phi_{t-\tau}^\gamma(\theta_\tau u, x) = (\theta_t u, \eta^\gamma(t-\tau, 0, \theta_\tau u)x) = (\theta_t u, \eta^\gamma(t,\tau,u)). \tag{2.16}$$

Define the function $\mathbf{Q} : \mathbb{R} \to C(\mathcal{U} \times \mathbb{R}, \mathcal{U} \times \mathbb{R})$ by

$$\mathbf{Q}(t) := \mathbf{T}\Phi_t^\gamma \circ \mathbf{P} \circ \mathbf{T}\Phi_{-t}^\gamma.$$

Then we have

$$\begin{aligned} \mathbf{Q}(t)(u,x) &= \mathbf{T}\Phi_t^\gamma \circ \mathbf{P} \circ \mathbf{T}\Phi_{-t}^\gamma(u,x) \\ &= \mathbf{T}\Phi_t^\gamma \circ \mathbf{P}(\theta_{-t}u, \eta^\gamma(-t,0,u)x) \\ &= \mathbf{T}\Phi_t^\gamma(\theta_{-t}u, \mathbf{P}_{\theta_{-t}u}\eta^\gamma(-t,0,u)x) \\ &= (u, \eta^\gamma(t,0,\theta_{-t}u)\mathbf{P}_{\theta_{-t}u}\eta^\gamma(-t,0,u)x) \end{aligned}$$

We define the mapping $Q : \mathbb{R} \times \mathcal{U} \to \mathbb{R}^d$ by

$$Q(t,u) := \eta^\gamma(t,0,\theta_{-t}u)\mathbf{P}_{\theta_{-t}u}\eta^\gamma(-t,0,u).$$

Now for $(u,x) \in \mathcal{U} \times \mathbb{R}^d$ we get

$$\begin{aligned} \mathbf{Q}(t) \circ \mathbf{T}\Phi_{t-\tau}^\gamma(u,x) &= \mathbf{T}\Phi_t^\gamma \circ \mathbf{P} \circ \mathbf{T}\Phi_{-t}^\gamma \circ \mathbf{T}\Phi_{t-\tau}^\gamma(u,x) \\ &= \mathbf{T}\Phi_t^\gamma \circ \mathbf{P} \circ \mathbf{T}\Phi_{-\tau}^\gamma(u,x) \\ &= \mathbf{T}\Phi_t^\gamma(\theta_{-\tau}u, \mathbf{P}_{\theta_{-\tau}u}\eta^\gamma(-\tau,0,u)x) \\ &= (\theta_{t-\tau}u, \eta^\gamma(t,0,\theta_{-\tau}u)\mathbf{P}_{\theta_{-\tau}u}\eta^\gamma(-\tau,0,u)x) \\ &= (\theta_{t-\tau}u, \eta^\gamma(t,\tau,\theta_{-\tau}u)\eta^\gamma(\tau,0,\theta_{-\tau}u)\mathbf{P}_{\theta_{-t}u}\eta^\gamma(-\tau,0,u)x) \\ &= (\theta_{t-\tau}u, \eta^\gamma(t,\tau,\theta_{-\tau}u)Q(\tau,u)x). \end{aligned}$$

On the other hand we have

$$\begin{aligned}\mathbf{Q}(t) \circ \mathbf{T}\Phi^{\gamma}_{t-\tau}(u,x) &= \mathbf{Q}(t)(\theta_{t-\tau}u, \eta^{\gamma}(t-\tau,0,u)x) \\ &= (\theta_{t-\tau}, Q(t,\theta_{t-\tau}u)\eta^{\gamma}(t-\tau,0,u)x) \\ &= (\theta_{t-\tau}, Q(t,\theta_{t-\tau}u)\eta^{\gamma}(t,\tau,\theta_{-\tau}u)x).\end{aligned}$$

Thus it follows

$$\eta^{\gamma}(t,\tau,\theta_{-\tau}u)Q(\tau,u) = Q(t,\theta_{t-\tau}u)\eta^{\gamma}(t,\tau,\theta_{-\tau}u).$$

Furthermore by definition we have $|(u,x)| := \|x\|$ for every $(u,x) \in \mathcal{U} \times \mathbb{R}$. We get

$$\begin{aligned}\|\eta^{\gamma}(t,\tau,\theta_{-\tau}u)Q(\tau,u)x\| &= \|\eta^{\gamma}(t-\tau,0,u)Q(\tau,u)x\| \\ &= |(\theta_{t-\tau}u, \eta^{\gamma}(t-\tau,0,u)Q(\tau,u)x| \\ &= |\mathbf{T}\Phi^{\gamma}_{t-\tau}(u, Q(\tau,u)x)| \\ &= |\mathbf{T}\Phi^{\gamma}_{t-\tau} \circ \mathbf{Q}(\tau)(u,x)| \\ &= |\mathbf{T}\Phi^{\gamma}_{t-\tau} \circ \mathbf{T}\Phi_{\tau} \circ \mathbf{P} \circ \mathbf{T}\Phi^{\gamma}_{-\tau}(u,x)| \\ &= |\mathbf{T}\Phi^{\gamma}_{t} \circ P \circ \mathbf{T}\Phi^{\gamma}_{-\tau}(u,x)|,\end{aligned}$$

and

$$\begin{aligned}\|\eta^{\gamma}(t,\tau,\theta_{-\tau}u)[\mathrm{id} - Q(\tau,u)]x\| &= \|\eta^{\gamma}(t-\tau,0,u)[\mathrm{id} - Q(\tau,u)]x\| \\ &= |(\theta_{t-\tau}u, \eta^{\gamma}(t-\tau,0,u)[\mathrm{id} - Q(\tau,u)]x| \\ &= |\mathbf{T}\Phi^{\gamma}_{t-\tau}(u, [\mathrm{id} - Q(\tau,u)]x)| \\ &= |\mathbf{T}\Phi^{\gamma}_{t-\tau} \circ [\mathrm{id} - \mathbf{Q}(\tau)](u,x)| \\ &= |\mathbf{T}\Phi^{\gamma}_{t-\tau} \circ [\mathrm{id} - \mathbf{T}\Phi^{\gamma}_{\tau} \circ \mathbf{P} \circ \mathbf{T}\Phi^{\gamma}_{-\tau}](u,x)| \\ &= |\mathbf{T}\Phi^{\gamma}_{t-\tau} \circ [\mathbf{T}\Phi^{\gamma}_{\tau} - \mathbf{T}\Phi^{\gamma}_{\tau} \circ \mathbf{P}] \circ \mathbf{T}\Phi^{\gamma}_{-\tau}(u,x)| \\ &= |\mathbf{T}\Phi^{\gamma}_{t} \circ [\mathrm{id} - \mathbf{P}] \circ \mathbf{T}\Phi^{\gamma}_{-t}(u,x)|.\end{aligned}$$

Thus if we set $v := \theta_{-\tau}u$ we obtain

$$\begin{array}{rclr}\eta^{\gamma}(t,\tau,v)Q(\tau,\theta_{\tau}v) & = & Q(t,\theta_t v)\eta^{\gamma}(t,\tau,v) & \text{for all } t,\tau \in \mathbb{R}, \\ \|\eta^{\gamma}(t,\tau,v)Q(\tau,\theta_{\tau}v)\| & \leq & Ke^{-\alpha(t-\tau)} & \text{for all } t \geq \tau, \\ \|\eta^{\gamma}(t,\tau,v)[\mathrm{id} - Q(\tau,\theta_{\tau}v)]\| & \leq & Ke^{\alpha(t-\tau)} & \text{for all } t \leq \tau.\end{array}$$

This means, that the linear differential equation $\dot{x} = [A_0 x + \sum_{i=1}^{m} v_i(t)A_i x - \gamma\,\mathrm{id}]x$ has an exponential dichotomy in the sense of Definition 6.1.2 with the constants α, K and the invariant projector $R : \mathbb{R} \to \mathcal{L}(\mathbb{R}^d)$ defined by

$$R(t) := Q(t, \theta_t v).$$

■

Before we come to the main result we have to introduce the following notation.

Definition 2.2.5 *Let $x \in \mathbb{R}^d$ and $N \subset \mathbb{R}^d$ with $x \in N$. Then we define the* reachable set in N *by*

$$\mathcal{O}_N^+(x) := \left\{ y \in N : \begin{array}{l} \text{there is a } u \in \mathcal{U} \text{ and a } T > 0 \text{ such that } \varphi(T,0,x,u) = y \\ \text{and } \varphi(t,0,x,u) \in N \text{ for all } t \in [0,T]. \end{array} \right\}.$$

Thus $\mathcal{O}_N^+(x)$ consists of all those points we can be reached from x without leaving N. If we have a control set $D \subset \mathbb{R}^d$ and a set $N \subset \mathbb{R}^d$ such that

$$x \in D \cap N \Rightarrow D \subset \mathcal{O}_N^+(x). \tag{2.17}$$

is fulfilled, it follows that $D \subset N$. On the other hand suppose, that there is a control set $D \subset \mathbb{R}^d$ and a set $N \subset \mathbb{R}^d$, such that (2.17) is not fulfilled. This means, that there are $x \in D \cap N$ and $y \in D$ such that x can only be steered to y by leaving N.

Theorem 2.2.6 *Consider the nonlinear control system*

$$\begin{array}{l} \dot{x} = f_0(x) + \sum_{i=1}^m u_i(t) f_i(x) \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ f.a.a. } t \in \mathbb{R}, \text{locally integrable}\} \end{array} \tag{2.18}$$

where U is a compact and convex subset of $\mathbb{R}^m$ containing 0 and $f_0, \dots, f_m$ are C^2 vector fields on $\mathbb{R}^d$. Suppose that the following conditions are satisfied.

(a) The nonlinear control system (2.11) has a singular point $x^* \in \mathbb{R}^d$ and there is an open neighborhood $V \subset \mathbb{R}^d$ of x^* such that the system (2.18) is locally accessible in $V \setminus \{x^*\}$.

(b) If we linearize the system (2.13) at the singular point x^*, then the corresponding linearized control flow $\mathbf{T}\Phi$ on $\mathcal{U} \times \mathbb{R}^d$ admits the decomposition

$$\mathcal{U} \times \mathbb{R} = \mathcal{V}^+ \oplus \mathcal{V}^- \tag{2.19}$$

into exponentially separated subbundles with

$$\sup \Sigma_{Ly}(\mathcal{V}^+) < 0 < \inf \Sigma_{Ly}(\mathcal{V}^-).$$

Then there is a neighborhood $N \subset \mathbb{R}^d$ of x^* such that there is no *set $D \subset \mathbb{R}^d$ with nonvoid interior and the property*

$$x \in D \cap N \Rightarrow D \subset \mathcal{O}_N^+(x).$$

Proof. For notational convenience we assume without loss of generality, that $x^* = 0$. Because the vector fields f_i of the nonlinear system (2.1) are C^2-vector fields, we can write

$$f_i(x) = A_i x + F_i(x)$$

where $A_i := \frac{\partial f_i}{\partial x}\Big|_{x=0}$ and $F_i(x)$ is a continuous differentiable vector field, with $\frac{\partial F_i}{\partial x}\Big|_{x=0} = 0$. Thus the system (2.18) can be written in the form

$$\dot{x} = A(u(t))x + F(t,x,u) \tag{2.20}$$

with

$$A(u(t))x := A_0 x + \textstyle\sum_{i=1}^m u_i(t) A_i x$$

and

$$F(t,x,u) := F_0(x) + \textstyle\sum_{i=1}^m u_i(t) F_i(x) \ .$$

Note, that $\frac{\partial F}{\partial x}(t,x,u)\big|_{x=0} = 0$ and $F(t,0,u) = 0$ for all $t \in \mathbb{R}$ and all $u \in \mathcal{U}$.

Step 1: By Lemma 2.2.3 we have

$$\Sigma_{Ly} = \Sigma_{dich}.$$

Thus by Lemma 2.2.4 it follows, that for every $u \in \mathcal{U}$ we have

$$\Sigma(A(u)) \subseteq \Sigma_{Ly}(\mathcal{V}^+) \cup \Sigma_{Ly}(\mathcal{V}^-), \tag{2.21}$$

and for every $\gamma \in (\sup \Sigma_{Ly}(\mathcal{V}^+), \inf \Sigma_{Ly}(\mathcal{V}^-))$ there is $\alpha \geq 0, K \geq 1$ such that for every $u \in \mathcal{U}$ the differential equation $\dot{x} = [A(u) - \gamma\,\mathrm{id}]x$ has a dichotomy with the constants α, K.

Step 2: According Proposition 6.1.9 for each $u \in \mathcal{U}$, there is a kinematic similarity transformation $S(u) : \mathbb{R} \to \mathcal{L}(\mathbb{R}^d)$ between

$$\dot{x} = A(u(t))x \tag{2.22}$$

and the decoupled linear differential equation

$$\dot{z} = A^{\pm}(u(t))z. \tag{2.23}$$

with

$$A^{\pm}(u(t)) := \mathrm{diag}\left(A^+(u(t)), A^-(u(t))\right)$$

such that

$$\Sigma(A^+(u)) \subseteq \Sigma_{Ly}(\mathcal{V}^+) \text{ and } \Sigma(A^-(u)) \subset \Sigma_{Ly}(\mathcal{V}^-) \tag{2.24}$$

because of (2.21). Because there are $\alpha > 0, K \geq 1$ such that the system (2.22) has for every $u \in \mathcal{U}$ a dichotomy with these constants (according to Step 1), we can conclude with Proposition 6.1.9 that there is a constant $\zeta > 0$ which is independent of u such that we have the estimate

$$\|S(t,u)\| \leq \zeta \text{ and } \left\|S^{-1}(t,u)\right\| \leq \zeta \text{ for all } t \in \mathbb{R}, u \in \mathcal{U}. \tag{2.25}$$

We denote the subspace corresponding to $A^-(u)$ by $X^-(u) < \mathbb{R}^d$ and the subspace corresponding to A^+ by $X^+(u) < \mathbb{R}^d$. We denote by $\|\cdot\|^{\pm,u}$ the norm on $X^+(u) \times X^-(u)$ defined by

$$\|(x,y)\|^{\pm,u} = \max\{\|x\|, \|y\|\}.$$

Then we have for every $(x,y) \in X^+(u) \times X^-(u)$

$$\|x+y\| \leq \|x\| + \|y\| \leq \max\{2\|x\|, 2\|y\|\} = 2\|(x,y)\|^{\pm,u} \tag{2.26}$$

Note that

$$\dim X^+(u) = \dim \mathcal{V}^+ \text{ and } \dim X^-(u) = \dim \mathcal{V}^-.$$

The projection of $\mathbb{R}^d$ onto $X^+(u)$ will be denoted by $P^+(u)$ and the projection of $\mathbb{R}^d$ onto $X^-(u)$ by $P^-(u)$. We denote the fundamental solution of (2.23) by $\eta^{\pm}(t,\tau,u)$, that of $\dot{x} = A^+(u(t))x$ by $\eta^+(t,\tau,u)$ and that of $\dot{y} = A^-(u(t))y$ by $\eta^-(t,\tau,u)$.

If we apply the kinematic similarity transformation $S(t,u)$ to the nonlinear control system (2.20) we get the following system on $X^+(u) \times X^-(u)$

$$\begin{aligned} \dot{x} &= A^+(u(t))x + F^+(t,x,y,u) \\ \dot{y} &= A^-(u(t))y + F^-(t,x,y,u) \end{aligned} \tag{2.27}$$

where we define

$$\begin{aligned} F^+(t,x,y,u) &:= P^+(u) \circ S(t,u) \circ F(t, S^{-1}(t,u)(x,y), u) \\ F^-(t,x,y,u) &:= P^-(u) \circ S(t,u) \circ F(t, S^{-1}(t,u)(x,y), u) \end{aligned}$$

We denote the solution of (2.27) by $\psi(t,\tau,x,y,u)$ for all $t,\tau \in \mathbb{R}$, $(x,y) \in X \times Y, u \in \mathcal{U}$.

Step 3: Now we choose $\alpha_i, \beta_i \in \mathbb{R}, i = 1,2$ with

$$\alpha_1 < \inf \Sigma_{Ly}(\mathcal{V}^+) < \sup \Sigma_{Ly}(\mathcal{V}^+) < \beta_1 < \alpha_2 < \inf \Sigma_{Ly}(\mathcal{V}^-) < \sup \Sigma_{Ly}(\mathcal{V}^+) < \beta_2$$

and $\delta > 0$ with

$$\delta < \frac{\alpha_2 - \beta_1}{2}.$$

Because of (2.24) for every $\gamma \in (-\infty, \alpha_1 - \delta) \cup (\beta_1 + \delta, \alpha_2 - \delta) \cup (\beta_2 + \delta, +\infty)$ there is a $K \geq 1$ and an $\alpha > 0$ such for all $u \in \mathcal{U}$ the linear differential equations

$$\dot{x} = [A(u(t)) - \gamma \operatorname{id}]x \tag{2.28}$$

have an exponential dichotomy with the constants K, α, i.e. if we denote by $\eta^\gamma(t,\tau,u)$ the fundamental solution of (2.28), then there is an invariant projector $Q^\gamma(\cdot,u)$ such that the dichotomy relation

$$\begin{aligned} \|\eta^\gamma(t,\tau,u)Q(\tau,u)\| &\leq Ke^{-a(t-\tau)} && \text{for } t \geq \tau, \\ \|\eta^\gamma(t,\tau,u)[\operatorname{id} - Q(\tau,u)]\| &\leq Ke^{a(t-s)} && \text{for } t \leq \tau, \end{aligned}$$

is fulfilled. According with Step 2 $\dot{x} = A(u(t))x$ and $\dot{z} = A^{\pm}(u(t))z$ are kinematic equivalent. Thus with Satz 3.26 and Lemma 3.19 in [29] it follows, that there are constants $K^{\pm} \geq 1$ and $\alpha > 0$ such that for $u \in \mathcal{U}$ the system $\dot{z} = [A^{\pm}(u(t)) - \gamma\,\mathrm{id}]z$ has also an exponential dichotomy with the invariant projector $Q^{\pm,\gamma}(t,u) := S^{-1}(t,u)Q^{\gamma}(t,u)S(t,u), t \in \mathbb{R}$ and the constants $K^{\pm}, \alpha$.

By Lemma 6.1.11, for all $u \in \mathcal{U}$ we get

$$
\begin{array}{lcll}
\|\eta^+(t,\tau,u)\|^{\pm,u} & \leq & K^{\pm}e^{\beta_1(t-\tau)} & \text{for all } t \geq \tau, \\
\|\eta^+(t,\tau,u)\|^{\pm,u} & \leq & K^{\pm}e^{\alpha_1(t-\tau)} & \text{for all } t \leq \tau, \\
\|\eta^-(t,\tau,u)\|^{\pm,u} & \leq & K^{\pm}e^{\beta_2(t-\tau)} & \text{for all } t \geq \tau, \\
\|\eta^-(t,\tau,u)\|^{\pm,u} & \leq & K^{\pm}e^{\alpha_2(t-\tau)} & \text{for all } t \leq \tau.
\end{array}
$$

Step 4: Now for every $u \in \mathcal{U}$ and every $\varepsilon > 0$ we define the *radial retraction* $r_\varepsilon : X^+(u) \times X^-(u) \to \mathrm{cl}\, B_\varepsilon(0)$ (with a slight abuse of notation) by

$$
r_\varepsilon(x,y,u) := \begin{cases} (x,y) & \text{for } \|x+y\| \leq \varepsilon, \\ \frac{\varepsilon}{\|x+y\|}(x,y) & \text{for } \|x+y\| > \varepsilon. \end{cases}
$$

Note, that $r_\varepsilon(x,y,u)$ has the global Lipschitz constant 1 for all $u \in \mathcal{U}$. We define the *reduced standard system*

$$
\begin{aligned}
\dot{x} &= A^+(u(t))x + F^+_\varepsilon(t,x,y,u) \\
\dot{y} &= A^-(u(t))y + F^-_\varepsilon(t,x,y,u)
\end{aligned} \tag{2.29}
$$

on $X^+(u) \times Y^-(u)$ by

$$
\begin{aligned}
F^+_\varepsilon(t,x,y,u) &:= F^+(t, r_\varepsilon(x,y,u), u), \\
F^-_\varepsilon(t,x,y,u) &:= F^-(t, r_\varepsilon(x,y,u), u),
\end{aligned} \tag{2.30}
$$

which coincides on $\mathbb{R}\times\, \mathrm{cl}\, B_\varepsilon(0)$ with the transformed system (2.27). This differential equation is in fact a standard system in the sense of Definition 6.1.10 of Appendix 6.1.2. The solutions starting at time τ at point $(x,y) \in X(u) \times Y(u)$ will be denoted by $\mu_\varepsilon(t,\tau,x,y,u)$ and they are unique and exist for all $t \in \mathbb{R}$.

Then for given $L, Q > 0$ there exists an $\varepsilon := \varepsilon(L,Q) > 0$ such that for all $u \in \mathcal{U}$ the nonlinearities (2.30) fulfill the properties

$$
\begin{array}{rcl}
\|F^+_\varepsilon(t,x,y,u) - F^+_\varepsilon(t,x',y',u)\|^{\pm,u} & \leq & L\,\|(x,y)-(x',y')\|^{\pm,u}\,, \\
\|F^-_\varepsilon(t,x,y,u) - F^-_\varepsilon(t,x',y',u)\|^{\pm,u} & \leq & L\,\|(x,y)-(x',y')\|^{\pm,u}\,, \\
\|F^+_\varepsilon(t,x,y,u)\|^{\pm,u} & \leq & Q, \\
\|F^-_\varepsilon(t,x,y,u)\|^{\pm,u} & \leq & Q,
\end{array} \tag{2.31}
$$

for all $(x,y),(x',y') \in X^+(u) \times X^-(u)$ and all $t \in \mathbb{R}$.

To see this, note first that

$$
F_i(0) = 0 \text{ and } \left.\frac{\partial F_i(x)}{\partial x}\right|_{x=0} = 0 \text{ for all } i = 0,\ldots,m.
$$

Thus by the mean value theorem there exists an $\varepsilon > 0$ such that for all $x, x' \in B_\varepsilon(0)$ and all $i = 1, \dots, m$ we have

$$\left\| F_i(x) - F_i(x') \right\| \le \frac{L}{2\zeta^2(1 + \operatorname{diam}(U))} \left\| x - x' \right\|,$$
$$\| F_i(x) \| \le \frac{Q}{2\zeta^2(1 + \operatorname{diam}(U))},$$

where ζ is given by (2.25). Then we get

$$\begin{aligned} \left\| F(x) - F(x') \right\| &\le \left\| F_0(x) - F_0(x') \right\| + \sum_{i=1}^{m} |u_i(t)| \left\| F_i(x) - F_i(x') \right\| \\ &\le \frac{L}{2\zeta^2(1 + \operatorname{diam}(U))} \left\| x - x' \right\| + \sum_{i=1}^{m} \frac{\operatorname{diam}(U) L}{2\zeta^2(1 + \operatorname{diam}(U))} \left\| x - x' \right\| \\ &= \frac{L}{2\zeta^2} \left\| x - x' \right\| \end{aligned}$$

and

$$\begin{aligned} \| F(x) \| &\le \| F_0(x) \| + \sum_{i=1}^{m} |u_i(t)| \, \| F_i(x) \| \\ &\le \frac{Q}{2\zeta^2}. \end{aligned}$$

Thus we have for all $u \in \mathcal{U}$ and $(x, y), (x', y') \in X^+(u) \times X^-(u)$ and all $t \in \mathbb{R}$

$$\begin{aligned} &\left\| F_\varepsilon^+(t, x, y, u) - F_\varepsilon^+(t, x', y', u) \right\|^{\pm,u} \\ &= \left\| P^+(u) \circ S(t, u)[F(t, S^{-1}(t, u) r_\varepsilon(x, y, u), u) - F(t, S^{-1}(t, u) r_\varepsilon(x', y', u), u)] \right\|^{\pm,u} \\ &\le \| S(t, u) \| \left\| F(t, S^{-1}(t, u) r_\varepsilon(x, y, u), u) - F(t, S^{-1}(t, u) r_\varepsilon(x', y', u), u) \right\| \\ &\le \frac{L}{2\zeta} \left\| S^{-1}(t, u)[r_\varepsilon(x, y, u) - r_\varepsilon(x', y'.u)] \right\| \\ &\le \frac{L}{2} \left\| r_\varepsilon(x, y, u) - r_\varepsilon(x', y', u) \right\| \\ &\le \frac{L}{2} \left\| (x, y) - (x', y') \right\| \\ &\le L \left\| (x, y) - (x', y') \right\|^{\pm,u} \end{aligned}$$

and

$$\begin{aligned} \left\| F_\varepsilon^+(t, x, y, u) \right\|^{\pm} &\le \frac{Q}{2\zeta} \left\| S^{-1}(t, u) r_\varepsilon(x, y)] \right\| \\ &\le \frac{Q}{2} \| r_\varepsilon(x, y)] \| \\ &\le Q \, \| (x, y) \|^{\pm,u}. \end{aligned}$$

Step 5: Choose $L > 0$ such that

$$KL < \frac{\delta}{K+1}$$

and choose $\varepsilon > 0$, such that the relation (2.31) holds for all $(x, y), (x', y') \in X^+(u) \times X^-(u), t \in \mathbb{R}$ and all $u \in \mathcal{U}$ and define

$$N := B_{\frac{\varepsilon}{\zeta}}(0).$$

Suppose, that there is a set $D \subset \mathbb{R}^d$ with nonvoid interior such that for all $x \in D$ it follows, that $D \subset \mathcal{O}_N^+(x)$. Now choose $p_0, p_1 \in D$. Because $p_1 \in D \subset \mathcal{O}_N^+(p_0)$ it follows, that there is a time $t_0 > 0$ and a control function $v_0 \in \mathcal{U}$ such that

$$\begin{aligned} &\varphi(t, 0, p_0, v_0) \in N \text{ for all } t \in [0, t_0] \text{ and} \\ &\varphi(t_0, 0, p_0, v_0) = p_1. \end{aligned}$$

In the same way one sees, that there are $t_1 > 0$ and $u_1 \in \mathcal{U}$ with

$$\begin{aligned} &\varphi(t, 0, p_1, v_1) \in N \qquad \text{for all } t \in [0, t_1] \text{ and} \\ &\varphi(t_1, 0, p_1, v_1) = p_0. \end{aligned}$$

Thus if we define the function $u \in \mathcal{U}$ by

$$u(t) := \begin{cases} v_0(t) & \text{for} \quad t \in [0, t_0], \\ v_1(t - t_0) & \text{for} \quad t \in [t_0, t_0 + t_1], \end{cases}$$

and continue it $\Theta := (t_0 + t_1)$-periodical, we get

$$\begin{aligned} &\varphi(t + k\Theta, 0, p_0, u) = \varphi(t, 0, p_0, u) && \text{and} \\ &\varphi(t, 0, p_0, u) \in B_{\frac{\varepsilon}{\zeta}}(0) && \text{for all } t \in \mathbb{R}, k \in \mathbb{Z}. \end{aligned}$$

Because $f_0(x) + \sum_{i=1}^m u_i(t) f_i(x)$ is Θ-periodic, there is a 2Θ-periodic similarity transformation $S(\cdot, u)$ between $\dot{x} = A(u(t))x$ and $\dot{z} = A^{\pm}(u(t))$, see Appendix 6.2 for the construction of $S(\cdot, u)$. This means, that for u the system (2.27) as well as the reduced system (2.29) are 2Θ-periodic in their t-components. Because $\varphi(t, 0, p_0, u) \in B_\varepsilon(0)$ it follows with $(x_0, y_0) := S(0, u)p_0$ that

$$\begin{aligned} \|\psi(t, 0, x_0, y_0, u)\| &= \|S(t, u)\varphi(t, 0, p_0, u)\| \\ &\leq \zeta \, \|\varphi(t, 0, p_0, u)\| \\ &< \varepsilon \end{aligned}$$

and we get therefore

$$\psi(t, 0, x_0, y_0, u) = \mu_\varepsilon(t, 0, x_0, y_0, u) \text{ for all } t \in \mathbb{R}.$$

We have chosen in Step 4 and 5 the constants $\varepsilon, \alpha, \beta, K, \delta$, such that we can apply the Hartman-Grobman Theorem 6.1.22 on the reduced system (2.29). By the Hartman-Grobman Theorem 6.1.22 the reduced standard system

$$\begin{aligned}\dot{x} &= A^{+}(u(t))x + F_{\varepsilon}^{+}(t,x,y,u)\\ \dot{y} &= A^{-}(u(t))y + F_{\varepsilon}^{-}(t,x,y,u)\end{aligned}$$

is topological equivalent (with respect to the trivial solution) to the linear system

$$\dot{z} = A^{\pm}(u(t))z$$

via a topologically equivalence $\mathcal{H} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ which is 2Θ-periodic. This means, that

$$\eta^{\pm}(t,0,u)\mathcal{H}(0,x_0,y_0) = \mathcal{H}(t,\mu_\varepsilon(t,0,x_0,y_0,u))$$

for all $t \in \mathbb{R}$. Because of periodicity we get

$$\begin{aligned}\eta^{\pm}(t+2\Theta k,0,u)\mathcal{H}(0,x_0,y_0) &= \mathcal{H}(t,\mu_\varepsilon(t+2\Theta k,0,x_0,y_0,u))\\ &= \mathcal{H}(t,\mu_\varepsilon(t,0,x_0,y_0,u))\\ &= \eta^{\pm}(t,0,u)\mathcal{H}(0,x_0,y_0)\end{aligned}$$

for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. Therefore $\eta^{\pm}(\cdot,0,u)\mathcal{H}(0,x_0,y_0)$ is 2Θ-periodic and it follows, that

$$\lambda(u_0, \mathcal{H}(0,x_0,y_0)) = 0.$$

But this is a contradiction to the assumption, that $0 \notin \Sigma_{Ly}$. ■

Remark 2.2.7 *There is an alternative proof for this theorem. By a cut-off technique we can restrict the linearized flow* $\mathbf{T}\Phi$*, such that we can apply a Hartman-Grobman Theorem on this restricted flow, cf. Bronstein and Kopanskii [7]. This means, that the restricted flow is topologically equivalent to a flow, which is generated by a blockdiagonal bilinear control system, like the system (2.23) in the proof the Theorem 2.2.6. Then we can make the same construction as in Step 5.*

Remark 2.2.8 *This Theorem does* not *show (as Theorem 2.1.4), that no control set with nonvoid interior intersects the neighborhood* N*. For example, there could be a family of periodic orbits, which do not lie completely in* N *but are arbitrarily close the singular point. Such effects can not be treated by linearization at the singular points* x^**, because this gives us only local results in a neighborhood around* x^**. The global behavior outside this neighborhood is not considered.*

Chapter 3

The Existence of Control Sets

In this chapter we will state a criterion, which yields the existence of control sets with nonvoid interior, such that the singular point of the control system lies in the closure of the control sets.

The idea to search for such a criterion was motivated by numerical experiments with the perturbed Duffing-van der Pol oscillator (cf. Chapter 5.1). This is a two dimensional system with singular point at the origin. For small perturbations, it has no control sets with nonvoid interior. But as the perturbation exceeds a certain level, then suddenly one obtains two control sets with nonvoid interior, such that the singular point lies in the closure of them.

The key to understand this behavior is the Lyapunov spectrum, which has only negative values for small perturbations as long as we can observe no control sets. If the perturbations are big enough, then 0 lies in the interior of the Lyapunov spectrum, and it has positive values, and we observe the two control sets with nonvoid interior.

Now the basic idea is to apply Proposition 1.1.21: If we find a pair $(u,p) \in \mathcal{U}\times\mathbb{R}^d$ such that the set $\{\varphi(t,0,p,u) : t \geq 0\}$ is bounded and some strong inner pair conditions are fulfilled, we can conclude that we get a control set with nonvoid interior which intersects $\pi_{\mathbb{R}^d}\omega(u,p)$. So in this chapter we will construct for certain $p \in \mathbb{R}^d$ a corresponding control function $u \in \mathcal{U}$ with these properties.

First we will explain the general construction idea. We need a periodic control function $u^s \in \mathcal{U}$, such that the corresponding linearized system has only negative Lyapunov exponents. Then we will need a periodic control function $u^h \in \mathcal{U}$ having the property, that the corresponding linearized system has positive and negative Lyapunov exponents (and no Lyapunov exponent 0). For the nonlinear ordinary differential equation, which is associated to u^h, we will define the local stable and unstable fibre bundles. Then we will define a subset of the unstable fibre bundle, called *target*, which is important for the construction of the control function u. We will characterize the local behavior of the system corresponding to u^h near the singular point. The construction of the control function $u \in \mathcal{U}$ will be made in a recursive manner. The resulting pair $(u,p) \in \mathcal{U} \times \mathbb{R}^d$ now has the property, that $\{\varphi(t,0,p,u) : t \geq 0\}$ is bounded. We will extract a pair $(u^*,p^*) \in \omega(u,x)$ which is also an element of the local unstable fibre bundle. Then the

Existence Theorem 3.7.1 shows, that we get a control set with nonvoid interior and the singular point in its closure. We will characterize this control set by the local unstable fibre bundle.

3.1 Basic Idea

We consider the control affine system on $\mathbb{R}^d$

$$\begin{array}{l} \dot{x} = f_0(x) + \sum_{i=1}^{m} u_i(t) f_i(x) \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\} \end{array} \tag{3.1}$$

where U is a compact and convex subset of $\mathbb{R}^m$. We assume that $f_0, \dots, f_m$ are C^2 vector fields on $\mathbb{R}^d$. Furthermore we suppose that for all $(u,x) \in \mathcal{U} \times \mathbb{R}^d$ the equation (3.1) has a unique solution $\varphi(t,\tau,x,u), t,\tau \in \mathbb{R}$, with $\varphi(\tau,\tau,x,u) = x$.

We suppose, that the system (3.1) has the singular point $x^* = 0 \in \mathbb{R}^d$. Note that the assumption $x^* = 0$ is no restriction. If our control affine system has a singular point $x^* \neq 0$ we can transform the control system by an affine transformation into a control affine system with 0 as singular point. The assumption $x^* = 0$ is made here only for notational convenience.

Associated with the nonlinear system (3.1) is the bilinear control system on $\mathbb{R}^d$:

$$\begin{array}{l} \dot{x} = A_0 x + \sum_{i=1}^{m} u_i(t) A_i x \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\} \end{array} \tag{3.2}$$

where $A_i := \left.\frac{\partial f_i}{\partial x}\right|_{x=0}$. For $u \in \mathcal{U}$ we denote the fundamental solution of (3.2) by $\eta(t,\tau,u)$, with $\eta(\tau,\tau,u)x = x$, where $\tau, t \in \mathbb{R}, x \in \mathbb{R}^d$.

For the rest of this chapter, we suppose, that the following conditions are satisfied.

Condition 3.1.1 *We assume, that there exist two periodic control functions $u^h, u^s \in \mathcal{U}$ with the following properties:*

(a) The control function u^h has period $\Theta \geq 0$ and the Lyapunov exponents $\lambda_1^h, \dots, \lambda_d^h$ of the corresponding linear system

$$\dot{x} = A_0 x + \sum_{i=1}^{m} u_i^h(t) A_i x$$

have the property

$$\lambda_1^h \geq \dots \geq \lambda_n^h > 0 > \lambda_{n+1}^h \geq \dots \geq \lambda_d^h \tag{3.3}$$

for a $n \in \{1, \dots, d-1\}$.

(b) For the control function u^s the corresponding linear system

$$\dot{x} = A_0 x + \sum_{i=1}^{m} u_i^s(t) A_i x$$

has the Lyapunov exponents $\lambda_1^s, \ldots, \lambda_d^s$ with the property

$$0 > \lambda_1^s \geq \lambda_2^s \geq \ldots \geq \lambda_d^s. \tag{3.4}$$

We call u^h the hyperbolic control function *and u^s the* stable control function.

The superscript ^{h} stands here for *hyperbolic* and the superscript ^{s} for *stable*. These symbols should emphasize, that if we apply the control function u^h to our control system, the linearized system has a hyperbolic structure, which is locally inherited by the nonlinear system. If we apply the control function u^s, then the nonlinear system is locally asymptotically stable at the singular point.

Now the question arises where we get this condition on the control system (3.1) from, and when is it fulfilled?

Consider the perturbed Duffing-van der Pol oscillator (cf. Chapter 5.1)

$$\begin{cases} \dot{x} = y \\ \dot{y} = (\alpha + u(t))x - \beta y - x^3 - x^2 y \end{cases} \tag{3.5}$$
$$u \in \mathcal{U}_\rho := \{u : \mathbb{R} \to \mathbb{R} : u(t) \in U_\rho \text{ for a.a. } t \in \mathbb{R}, \text{ locally integrable}\}$$

with $U_\rho = [-\rho, \rho] \subset \mathbb{R}$. Then, for $\rho < \frac{1}{4}$, we investigate numerically, that there is no control set with nonvoid interior. For $\rho > \frac{1}{4}$ suddenly we get two control sets with nonvoid interior, such that the singular point lies in the closure of the both control sets. As long as $\rho \in [0, \frac{1}{4})$ there are two subbundles $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{U} \times \mathbb{R}^d$ with

$$\Sigma_{Ly} = \Sigma_{Ly}(\mathcal{V}_1) \cup \Sigma_{Ly}(\mathcal{V}_2) \subset \mathbb{R}^-.$$

Thus, for every control function the corresponding Lyapunov exponents are negative. For $\rho \in (\frac{1}{4}, \frac{3}{4})$ we compute that $0 \in \operatorname{int} \Sigma_{Ly}(\mathcal{V}_2)$. The Duffing-van der Pol oscillator is a two-dimensional system, thus it follows $\operatorname{cl} \Sigma_{Fl} = \Sigma_{Ly}$ (cf. Theorem 1.2.12). Because of $\dim \mathcal{V}_2 = 1$ we can find a periodic control function u^h with Lyapunov exponents as in (3.3).

Now for a general system (3.1) we may frequently assume, that $\Sigma_{Ly} = \operatorname{cl} \Sigma_{Fl}$ (cf. Theorem 1.2.11). If we now have the decomposition $\mathcal{U} \times \mathbb{R}^d = \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_l$ for exponentially separated subbundles $\mathcal{V}_1, \ldots, \mathcal{V}_l$ (cf. Theorem 1.2.7) and

$$\Sigma_{Ly}(\mathcal{V}_i) \subset \mathbb{R}^- \text{ and } 0 \in \operatorname{int} \Sigma_{Ly}(\mathcal{V}_l) \text{ with } \dim \mathcal{V}_l = 1$$

then we can find control functions u^h, u^s as in Condition 3.1.1.

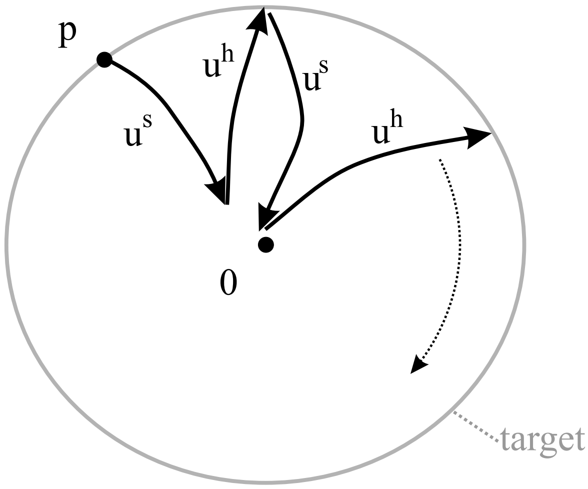

Figure 3.1: Basic construction idea of the control function u.

But if $\dim \mathcal{V}_l > 1$ then it is not guaranteed, if we can find a control function u^s. It could be, that for every $u \in \mathcal{U}$ we find an $x \in \mathbb{R}^d$ with $\lambda(u, x) > 0$. The same happens, if we have $0 \in \operatorname{int} \Sigma_{Ly}(\mathcal{V}_j)$ for $j \neq l$. Thus in these cases, the Lyapunov spectrum does not indicate the existence of the two control functions u^h and u^s.

It is not always a hopeless mission, to find u^h and u^s. In Chapter 5.2 we show by the example of the perturbed Lorenz equation, that it sometimes suffices to look at constant control functions. Then, by calculating the eigenvalues of the corresponding linearized spectrum, one can possibly find constant u^h and $u^s \in U$.

To obtain the existence of control sets with nonvoid interior under the Condition 3.1.1, the basic idea is to apply Proposition 1.1.21. Thus we need a pair $(u, p) \in \mathcal{U} \times \mathbb{R}^d$ such that $\{\varphi(t, 0, p, u) : t \geq 0\}$ is bounded. We will construct such a control function as follows. First we choose a starting point $p \in \mathbb{R}^d$, close enough to the singular point 0, such that we can characterize the qualitative behavior of the ordinary differential equations corresponding to u^h and u^s. The system corresponding to u^s is locally asymptotic stable. For u^h we get local stable and unstable fibre bundles, which means that if we start with a point near the singular point 0, the solutions are driven away from 0. The idea is to construct u by switching between u^h and u^s (cf. Figure 3.1).

We steer some time towards the origin with u^s. Then we switch to u^h and get driven away from the singular point 0, until we reach some set, the so called target set. Then we steer towards the origin with u^s and so forth. By choosing the appropriate switching times, we can achieve, that $\pi_{\mathbb{R}^d}\omega(u, p) \neq \{0\}$.

3.2 The Hyperbolic System

In this section we will have a closer look at the *hyperbolic* system

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i^h(t) f_i(x). \tag{3.6}$$

We want to define the local stable and unstable fibre bundles for this system and define the target set (cf. Figure 3.1). We refer here to Appendix 6.2 for the technical background of the following results.

Because the vector fields f_i are C^2-vector fields, we can write

$$f_i(x) = A_i x + F_i(x)$$

where $F_i(x)$ is a continuous differentiable vector field, with $\frac{\partial F_i}{\partial x}\Big|_{x=0} = 0$. Thus system (3.6) can be written in the form

$$\dot{x} = A(t)x + F(t,x) \tag{3.7}$$

with

$$A(t)x := A_0 x + \sum_{i=1}^{m} u_i^h(t) A_i x$$

and

$$F(t,x) := F_0(x) + \sum_{i=1}^{m} u_i^h(t) F_i(x).$$

Note, that $A(\cdot)$ as well as $F(\cdot, x)$ are Θ-periodic because u^h is Θ-periodic, and that $\frac{\partial F}{\partial x}(t,x)\big|_{x=0} = 0$ and $F(t,0) = 0$ for all $t \in \mathbb{R}$. Thus equation (3.7) has the form of equation (6.22).

The associated *linearized* system

$$\dot{x} = A(t)x \tag{3.8}$$

has the fundamental solution $\eta(t, \tau, u^h)$. As in Chapter 6.2 there is a 2Θ-periodic linear transformation $g(t)$, defined as in (6.24), which transforms the nonautonomous system (3.8) into the autonomous system

$$\dot{x} = Rx$$

with an $R \in \mathcal{L}(\mathbb{R}^d)$. If we arrange the eigenvalues $\xi_1, \dots, \xi_d$ in the right way we have $\lambda_i = \mathrm{Re}(\xi_i)$. Let X denote the sum of the generalized eigenspaces of R which belong to the eigenvalues $\xi_1, \dots \xi_n$ and Y the sum of the generalized eigenspaces of the eigenvalues $\xi_{n+1}, \dots, \xi_d$.

The projection P onto $X \times Y$ is defined by

$$P: \begin{array}{ccc} \mathbb{R}^d & \to & X \times Y \\ z & \mapsto & (P_X z, P_Y z) \end{array}$$

with the linear projections $P_X : \mathbb{R}^d \to X$ and $P_Y : \mathbb{R}^d \to Y$ onto X and Y.

With help of these projections we define the following blockdiagonal linear system on $X \times Y$

$$\begin{aligned} \dot{x} &= A^+ x \\ \dot{y} &= A^- y \end{aligned} \tag{3.9}$$

with $A^+ := P_X R P_X^{-1}$ and $A^- := P_Y R P_Y^{-1}$. The fundamental solution of equation (3.9) will be denoted by

$$\nu(t, \tau, u^h) = \operatorname{diag}(\nu_X(t, \tau, u^h), \nu_Y(t, \tau, u^h))$$

Define the transformation

$$\mathcal{F}: \begin{array}{ccc} \mathbb{R} \times \mathbb{R}^d & \to & X \times Y \\ (t, x) & \mapsto & P \circ g^{-1}(t) x \end{array}$$

With the transformation $\mathcal{F}$, the original system (3.6) gets transformed into following differential equation on $X \times Y$

$$\begin{aligned} \dot{x} &= A^+ x + F^+(t, x, y) \\ \dot{y} &= A^- y + F^-(t, x, y) \end{aligned} \tag{3.10}$$

which we call the *transformed* system, where F^+ and F^- are defined as in (6.30) and (6.31). We denote the solutions of (3.10) by $\psi(t, \tau, x, y, u^h)$ with $\psi(\tau, \tau, x, y, u^h) = (x, y)^T$.

Finally, by defining F_ε^+ and F_ε^- as in (6.30) and (6.31), we get for every $\varepsilon > 0$ the *reduced standard system*

$$\begin{aligned} \dot{x} &= Ax + F_\varepsilon^+(t, x, y) \\ \dot{y} &= By + F_\varepsilon^-(t, x, y) \end{aligned} \tag{3.11}$$

on $X \times Y$ which coincides on $\mathbb{R} \times \operatorname{cl} B_\varepsilon(0)$ with the transformed system (3.10). We denote the solutions of (3.11) by $\mu_\varepsilon(t, \tau, x, y, u^h)$ with $\mu_\varepsilon(\tau, \tau, x, y, u^h) = (x, y)^T$.

Definition 3.2.1 *We define the mapping*

$$\varepsilon(\cdot) : [0, \infty) \to [0, \infty)$$

as in Remark 6.2.5. After choosing α, β, δ and K as in (6.35), (6.38) and (6.39) we define L^* as in (6.40) and $\varepsilon^* := \varepsilon(L^*)$.

L^* is a Lipschitz constant for the right hand side of (3.11), for which the reduced system (3.11) fulfills the existence properties of the Existence Theorem 6.1.13 on invariant fibre bundles. The map $\varepsilon(\cdot) : [0,\infty) \to [0,\infty)$ maps every $L \in [0, L^*]$ to an $\varepsilon(L)$ such that the corresponding reduced system (3.11) has Lipschitz constant L. Thus for every $\varepsilon \in (0, \varepsilon^*)$ we get (globally) stable and unstable fibre bundles for the reduced system.

Note that we have chosen α, β, δ, K such that $0 \in (\beta + KL, \alpha - KL)$.

For $\varepsilon \in (0, \varepsilon^*]$ we define

- the *unstable* fibre bundle $\mathcal{X}_\varepsilon \subset \mathbb{R} \times X \times Y$ which is characterized by

$$\begin{aligned}\mathcal{X}_\varepsilon &= \{(\tau, x, y) \in \mathbb{R} \times X \times Y : \mu_\varepsilon(\cdot, \tau, x, y, u^h) \text{ is } \gamma^-\text{-quasibounded}\} \\ &= \{(\tau, x, y) \in \mathbb{R} \times X \times Y : y = w_\varepsilon^+(\tau, x)\}\end{aligned}$$

  for every $\gamma \in (\beta + KL, \alpha - KL)$ with corresponding mapping $w_\varepsilon^+ : \mathbb{R} \times X \to Y$.

- the *stable* fibre bundle $\mathcal{Y}_\varepsilon \subset \mathbb{R} \times X \times Y$ which is characterized by

$$\begin{aligned}\mathcal{Y}_\varepsilon &= \{(\tau, x, y) \in \mathbb{R} \times X \times Y : \mu_\varepsilon(\cdot, \tau, x, y, u^h) \text{ is } \gamma^+\text{-quasibounded}\} \\ &= \{(\tau, x, y) \in \mathbb{R} \times X \times Y : x = w_\varepsilon^-(\tau, y)\}\end{aligned}$$

  for every $\gamma \in (\beta + KL, \alpha - KL)$ with corresponding mapping $w_\varepsilon^- : \mathbb{R} \times Y \to X$.

- the *local unstable* fibre bundle $\mathcal{X}_\varepsilon^{loc} \subset \mathbb{R} \times \mathbb{R}^d$ by

$$\mathcal{X}_\varepsilon^{loc} := \{(\tau, p) \in \mathbb{R} \times \mathbb{R}^d : (\tau, \mathcal{F}(\tau)p) \in \mathcal{X}_\varepsilon\}.$$

- the *local stable* fibre bundle $\mathcal{Y}_\varepsilon^{loc}(u^h) \subset \mathbb{R} \times \mathbb{R}^d$ by

$$\mathcal{Y}_\varepsilon^{loc} := \{(\tau, p) \in \mathbb{R} \times \mathbb{R}^d : (\tau, \mathcal{F}(\tau)p) \in \mathcal{Y}_\varepsilon\}.$$

Remark 3.2.2 *Note, that $\mathcal{X}_\varepsilon, \mathcal{Y}_\varepsilon, \mathcal{X}_\varepsilon^{loc}$ and $\mathcal{Y}_\varepsilon^{loc}$ are uniquely defined and do not depend on α, β, K, δ chosen by Remark 6.1.14.*

Because the reduced standard system (3.11) fulfills the conditions of Theorem 6.1.18, we may also apply Theorem 6.1.20, which yields the existence of asymptotic phases . Denote by

$$\begin{array}{rccc}\hat{\mathcal{P}}_\varepsilon : & \mathbb{R} \times X \times Y & \to & \mathcal{X}_\varepsilon \\ & (t, x, y) & \mapsto & (t, \mathcal{P}_\varepsilon(t, x, y))\end{array} \tag{3.12}$$

the asymptotic phase of $\mathcal{X}_\varepsilon$ with the corresponding function $\mathcal{P}_\varepsilon : \mathbb{R} \times X \times Y \mapsto X \times Y$. For the hyperbolic system (3.6), we define a local version $\mathcal{P}_\varepsilon^{loc}$ of $\mathcal{P}_\varepsilon$ by

$$\begin{array}{rccc}\hat{\mathcal{P}}_\varepsilon^{loc} : & \mathbb{R} \times \mathbb{R}^d & \to & \mathcal{X}_\varepsilon^{loc}(u^h) \\ & (t, p) & \mapsto & (t, \mathcal{F}^{-1}(t)\mathcal{P}_\varepsilon(t, \mathcal{F}(t)p)).\end{array} \tag{3.13}$$

which we denote as *local asymptotic phase* . We define

$$\mathcal{P}_\varepsilon^{loc}(t,p) := \mathcal{F}^{-1}(t)\mathcal{P}_\varepsilon(t,\mathcal{F}(t)p).$$

The functions $\hat{\mathcal{P}}_\varepsilon$ and $\hat{\mathcal{P}}_\varepsilon^{loc}$ are 2Θ-periodic in the t-component and $\mathcal{P}_\varepsilon(t,0) = 0$ and $\mathcal{P}_\varepsilon^{loc}(t,0) = 0$ for all $t \in \mathbb{R}$.

For $\varepsilon \in (0,\varepsilon^*]$ we can also apply the nonautonomous Hartman-Grobman Theorem 6.1.22 to the reduced system. Then we obtain a topological equivalence between the reduced system (3.11) and the linear system (3.8)

$$\mathcal{H}_\varepsilon : \mathbb{R} \times X \times Y \to X \times Y.$$

For the hyperbolic system (3.6) we define $\mathcal{H}_\varepsilon^{loc} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$\mathcal{H}_\varepsilon^{loc}(t,p) := \mathcal{F}^{-1}(t)\mathcal{H}_\varepsilon(t,\mathcal{F}(t)p).$$

Next we define subsets of the unstable manifold $\mathcal{X}_\varepsilon^{loc}$ for the reduced system (3.11):

Definition 3.2.3 *For every $\varepsilon \in (0,\varepsilon^*]$ and every $\rho > 0$ we define*

$$\mathcal{X}_{\varepsilon,\leq\rho} := \{(t,x,y) \in \mathcal{X}_\varepsilon : \|(x,y)\| \leq \rho\}$$

and

$$\mathcal{X}_{\varepsilon,\leq\rho}(t) := \{(x,y) \in \mathcal{X}_\varepsilon : (t,x,y) \in \mathcal{X}_{\varepsilon,\leq\rho}\}$$

Note, that for every $t \in \mathbb{R}$ we have $\mathcal{X}_{\varepsilon,\leq\rho}(t) = \mathcal{X}_\varepsilon(t) \cap \operatorname{cl} B_\rho(0)$ and since $w_\varepsilon^+(\cdot,x)$ is 2Θ-periodic, we have

$$\mathcal{X}_{\varepsilon,\leq\rho}(t + 2k\Theta) = \mathcal{X}_{\varepsilon,\leq\rho}(t)$$

for all $t \in \mathbb{R}$ and all $k \in \mathbb{Z}$.

Lemma 3.2.4 *For every $\varepsilon \in (0,\varepsilon^*], \rho > 0$ and every $a,b \in \mathbb{R}$ with $-\infty < a \leq b < \infty$ the set $\mathcal{X}_{\varepsilon,\leq\rho}([a,b]) := \bigcup_{t\in[a,b]} \mathcal{X}_{\varepsilon,\leq\rho}(t)$ is compact.*

Proof. Let $(x_n, w_\varepsilon^+(t_n,x_n))_{n\in\mathbb{N}} \subset \mathcal{X}_{\varepsilon,\leq\rho}([a,b])$ be a sequence with $(t_n)_{n\in\mathbb{N}} \subset [a,b]$ and $(x_n)_{n\in\mathbb{N}} \in X$. Because $[a,b]$ is compact, there exists a convergent subsequence of (t_n) (which we will denote for abbreviation again as (t_n)) with $\lim_{n\to\infty} t_n = t^* \in [a,b]$. Since $\|(x_n, w_\varepsilon^+(t_n,x_n))\| \leq \rho$ there is a convergent subsequence (which we denote for abbreviation $(x_n, w_\varepsilon^+(t_n,x_n))_{n\in\mathbb{N}}$) with $\lim_{n\to\infty}(x_n, w_\varepsilon^+(t_n,x_n)) = (x^*,y^*)$ and $\|(x^*,y^*)\| \leq \rho$. Because of continuity of w_ε^+ we get

$$\lim_{n\to\infty}(x_n, w_\varepsilon^+(t_n,x_n)) = (x^*, w_\varepsilon^+(t^*,x^*)) \in \mathcal{X}_\varepsilon([a,b]).$$

Thus $(x^*,y^*) \in \mathcal{X}_{\varepsilon,\leq\rho}([a,b])$, which shows that $\mathcal{X}_{\varepsilon,\leq\rho}([a,b])$ is compact. ■

For the construction of the control function u we have to define the set, to which we want to steer with control function u^h (cf. Figure 3.1). We call this set the *target set* and it will be a subset of $\mathcal{X}_\varepsilon^{loc}$. For the linear system, the target T_ρ is just the sphere with radius ρ in X. Then we will extend this definition to the level of the reduced system (3.11) and then to the level of the hyperbolic system (3.6).

Definition 3.2.5 *For $\rho > 0$ define the* sphere T_ρ in X with radius ρ *by*

$$T_\rho := \{(x, 0) \in X \times Y : \|x\| = \rho\},$$

and the disc D_ρ in X with radius ρ *by*

$$D_\rho := \{(x, 0) \in X \times Y : \|x\| \le \rho\}.$$

Definition 3.2.6 *For every $\varepsilon \in (0, \varepsilon^*]$ and for every $\rho > 0$ we define the target $\mathcal{T}_{\varepsilon,\rho}$ and the target disc $\mathcal{D}_{\varepsilon,\rho}$ by*

$$\begin{aligned} \mathcal{T}_{\varepsilon,\rho} &:= \{(t, x, w_\varepsilon^+(t, x)) \in \mathcal{X}_\varepsilon : (x, w_\varepsilon^+(t, x)) \in \mathcal{H}_\varepsilon^{-1}(t, T_\rho)\} \\ \mathcal{D}_{\varepsilon,\rho} &:= \{(t, x, w_\varepsilon^+(t, x)) \in \mathcal{X}_\varepsilon : (x, w_\varepsilon^+(t, x)) \in \mathcal{H}_\varepsilon^{-1}(t, D_\rho)\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{T}_{\varepsilon,\rho}(t) &:= \{(x, y) \in \mathcal{T}_\varepsilon(t) : (t, x, y) \in \mathcal{T}_{\varepsilon,\rho}\} \\ \mathcal{D}_{\varepsilon,\rho}(t) &:= \{(x, y) \in \mathcal{T}_\varepsilon(t) : (t, x, y) \in \mathcal{T}_{\varepsilon,\rho}\} \end{aligned}$$

for all $t \in \mathbb{R}$.

Note, that since $\mathcal{H}_\varepsilon$ is 2Θ-periodic, we have

$$\begin{aligned} \mathcal{T}_{\varepsilon,\rho}(t + 2k\Theta) &= \mathcal{T}_{\varepsilon,\rho}(t) \\ \mathcal{D}_{\varepsilon,\rho}(t + 2k\Theta) &= \mathcal{D}_{\varepsilon,\rho}(t) \end{aligned}$$

for all $t \in \mathbb{R}$ and all $k \in \mathbb{Z}$.

Lemma 3.2.7 *For every $\varepsilon \in (0, \varepsilon^*], \rho > 0$ and every $a, b \in \mathbb{R}$ with $-\infty < a \le b < \infty$ the sets $\mathcal{T}_{\varepsilon,\rho}([a, b]) := \bigcup_{t\in[a,b]} \mathcal{T}_{\varepsilon,\rho}(t)$ and $\mathcal{D}_{\varepsilon,\rho}([a, b]) := \bigcup_{t\in[a,b]} \mathcal{D}_{\varepsilon,\rho}(t)$ are compact. Furthermore, $\mathcal{D}_{\varepsilon,\rho}([a, b])$ is pathconnected. If $n = 1$, then $\mathcal{T}_{\varepsilon,\rho}([a, b])$ consists of two continuous curves, and if $d \ge 2$, then $\mathcal{T}_{\varepsilon,\rho}([a, b])$ is pathconnected, too.*

Proof. First we show that $\mathcal{T}_{\varepsilon,\rho}([a, b])$ is compact. Let $(t_n, x_n, y_n)_{n\in\mathbb{N}} \subset \mathcal{T}_{\varepsilon,\rho}$ be a sequence with $t_n \in [a, b]$. Since $[a, b]$ is compact there is a convergent subsequence of $(t_n)_{n\in\mathbb{N}}$ which we denote again by $(t_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} t_n = t \in [a, b]$. The sequence $(p_n, q_n) := \mathcal{H}_\varepsilon(t_n, x_n, y_n) \in T_\rho$ has a convergent subsequence which we denote again by $(p_n, q_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}(p_n, q_n) = (p, q) \in T_\rho$. Thus we get

$$\lim_{n\to\infty} (t_n, x_n, y_n) = \lim_{n\to\infty} (t_n, \mathcal{H}^{-1}(t_n, p_n, q_n) = (t, \mathcal{H}^{-1}(p, q)) \in \mathcal{T}_{\varepsilon,\rho}([a, b])$$

which shows that $\mathcal{T}_{\varepsilon,\rho}([a,b])$ is compact. The compactness of $\mathcal{D}_{\varepsilon,\rho}([a,b])$ is shown similarly.

If $n=1$, then $\dim X = 1$. Thus $T_\rho = \{(\rho,0),(-\rho,0)\}$ and by definition of $\mathcal{T}_{\varepsilon,\rho}([a,b])$ we have

$$\mathcal{T}_{\varepsilon,\rho}([a,b]) = \{\mathcal{H}^{-1}(t,\rho,0) : t\in[a,b]\} \cup \{\mathcal{H}^{-1}(t,-\rho,0) : t\in[a,b]\}.$$

To show that $\mathcal{T}_{\varepsilon,\rho}([a,b])$ is pathconnected for $d\geq 2$, take two points $p,q\in\mathcal{T}_{\varepsilon,\rho}([a,b])$. Then there are $t_p,t_q\in[a,b]$ with $(t_p,p),(t_q,q)\in\mathcal{T}_{\varepsilon,\rho}([a,b])$. Define $(x,0):=\mathcal{H}_\varepsilon(t_p,p)\in T_\rho$ and $(y,0):=\mathcal{H}_\varepsilon(t_q,q)\in T_\rho$. Because $[a,b]\times T_\rho$ is pathconnected there is a continuous path $\tilde{c}:[0,1]\to[a,b]\times T_\rho$ with $\tilde{c}(0)=(t_p,x,0)$ and $\tilde{c}(1)=(t_q,y,0)$. Thus the continuous path $c:[0,1]\to\mathcal{T}_{\varepsilon,\rho}([a,b])$ defined by $c(t):=\mathcal{H}_\varepsilon^{-1}(t,\tilde{c}(t))$ has the property $c(0)=(t_p,p)$ and $c(1)=(t_q,q)$. Therefore the set $\mathcal{T}_{\varepsilon,\rho}([a,b])$ is pathconnected. Similarly one can prove that $\mathcal{D}_{\varepsilon,\rho}([a,b])$ is pathconnected. ■

The next lemma shows, that for every open neighborhood of 0 we can choose a $\rho>0$ such that for every $t\in\mathbb{R}$ the disk $\mathcal{D}_{\varepsilon,\rho}(t)$ is contained in the ball. Note that this statement is not totally obvious, because $\mathcal{D}_{\varepsilon,\rho}$ is defined by the topological conjugation $\mathcal{H}_\varepsilon$.

Lemma 3.2.8 *For every $\varepsilon\in(0,\varepsilon^*]$ and every $\sigma>0$ there exists an $\rho>0$ such*

$$\mathcal{D}_{\varepsilon,\rho}\subset\mathcal{X}_{\varepsilon,\leq\sigma}.$$

Proof. Because $\mathcal{D}_{\varepsilon,\rho}(\cdot)$ is 2Θ-periodic, it suffices to show, that there is a $\rho>0$ with $\mathcal{D}_{\varepsilon,\rho}(t)\subset\mathcal{X}_{\varepsilon,\leq\sigma}(t)$ for all $t\in[0,2\Theta]$, which means, that $\max_{t\in[0,2\Theta]}\|\mathcal{D}_{\varepsilon,\rho}(t)\|<\sigma$.

The mapping

$$\begin{array}{rccc} m: & [0,2\Theta]\times X\times Y & \longrightarrow & \mathbb{R} \\ & (t,x,y) & \longmapsto & \left\|\mathcal{H}^{-1}(t,x,y)\right\| \end{array}$$

is continuous with $m(t,0,0)=0$ for all $t\in[0,2\Theta]$. Therefore, for each $t\in[0,2\Theta]$ there is a neighborhood $V_t\subset\mathbb{R}$ of t and a $\rho(t)>0$ with

$$\sup\{m(s,x,y) : s\in V_t,(x,y)\in\operatorname{cl}B_{\rho(t)}(0)\}<\sigma.$$

Because $[0,2\Theta]$ is compact there is a $\rho>0$ with

$$\sup\{m(s,x,y) : s\in[0,2\Theta],(x,y)\in\operatorname{cl}B_{\rho}(0)\}<\sigma.$$

Thus for all $t\in[0,2\Theta]$ and all $(x,0)\in\operatorname{cl}B_\sigma(0)$ we have $\left\|\mathcal{H}^{-1}(t,x,0)\right\|<\sigma$, which means, that $\|\mathcal{D}_{\varepsilon,\rho}(t)\|<\sigma$. ■

Finally we define the target for the hyperbolic system (3.6).

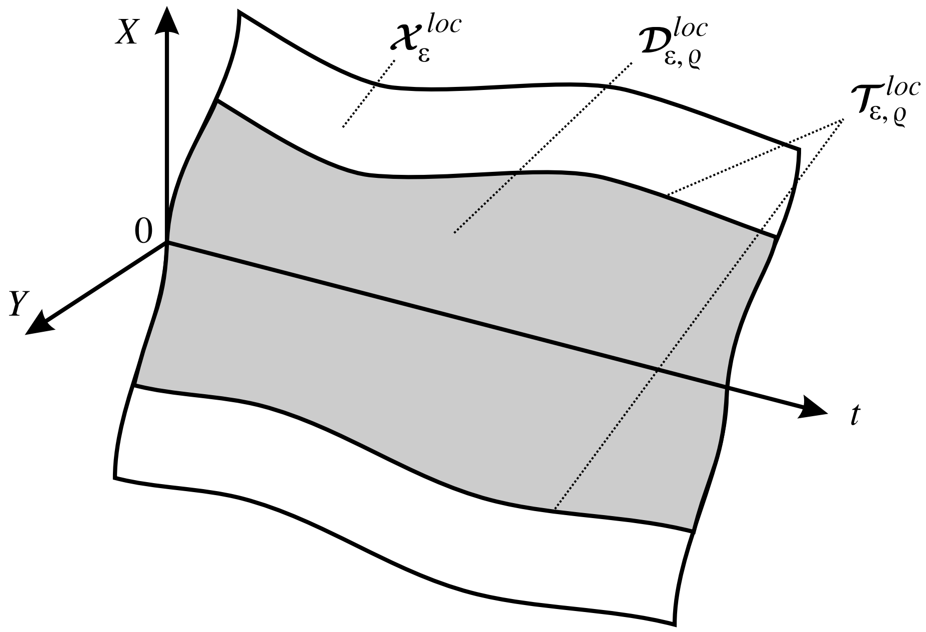

Figure 3.2: The target $\mathcal{T}_{\varepsilon,\rho}^{loc}$ and the target disc $\mathcal{D}_{\varepsilon,\rho}^{loc}$.

Definition 3.2.9 *For every $\varepsilon \in (0, \varepsilon^*]$ and every $\rho > 0$ we define the* target $\mathcal{T}_{\varepsilon,\rho}^{loc}$ *and the* target disc $\mathcal{D}_{\varepsilon,\rho}^{loc}$

$$\mathcal{T}_{\varepsilon,\rho}^{loc} := \{(t,p) \in \mathbb{R} \times \mathbb{R}^d : (t, \mathcal{F}(t)p) \in \mathcal{T}_{\varepsilon,\rho}\}$$
$$\mathcal{D}_{\varepsilon,\rho}^{loc} := \{(t,p) \in \mathbb{R} \times \mathbb{R}^d : (t, \mathcal{F}(t)p) \in \mathcal{D}_{\varepsilon,\rho}\}$$

and

$$\mathcal{T}_{\varepsilon,\rho}^{loc}(t) := \{x \in \mathbb{R}^d : (t,x) \in \mathcal{T}_{\varepsilon,\rho}^{loc}\}$$
$$\mathcal{D}_{\varepsilon,\rho}^{loc}(t) := \{x \in \mathbb{R}^d : (t,x) \in \mathcal{D}_{\varepsilon,\rho}^{loc}\}$$

for all $t \in \mathbb{R}$.

For $\dim X = 1$, the target $\mathcal{T}_{\varepsilon,\rho}^{loc}$ and the target disc $\mathcal{D}_{\varepsilon,\rho}^{loc}$ are sketched in Figure 3.2. Because $\mathcal{F}$ is 2Θ-periodic, we have

$$\mathcal{T}_{\varepsilon,\rho}^{loc}(t + 2k\Theta) = \mathcal{T}_{\varepsilon,\rho}(t)$$
$$\mathcal{D}_{\varepsilon,\rho}^{loc}(t + 2k\Theta) = \mathcal{D}_{\varepsilon,\rho}(t)$$

for all $t \in \mathbb{R}$ and all $k \in \mathbb{Z}$.

Notation 3.2.10 *According to Lemma 6.2.9 there is an $\hat{\varepsilon} \in (0, \varepsilon^*]$ such that for every $\varepsilon \in (0, \hat{\varepsilon}]$ there is a neighborhood $W(\varepsilon) \subset \mathbb{R}^d$ of 0 such that for all $p \in \mathcal{X}_\varepsilon^{loc}(\tau) \cap W(\varepsilon)$*

and all $q \in \mathcal{Y}_\varepsilon^{loc}(\tau) \cap W(\varepsilon)$ *we have*

$$\varphi(t,\tau,p,u^h) = \mathcal{F}^{-1}(t)\psi(t,\tau,\mathcal{F}(\tau)p,u^h) = \mathcal{F}^{-1}(t)\mu_\varepsilon(t,\tau,\mathcal{F}(\tau)p,u^h) \text{ for all } t \leq \tau,$$
$$\varphi(t,\tau,q,u^h) = \mathcal{F}^{-1}(t)\psi(t,\tau,\mathcal{F}(\tau)q,u^h) = \mathcal{F}^{-1}(t)\mu_\varepsilon(t,\tau,\mathcal{F}(\tau)q,u^h) \text{ for all } t \geq \tau.$$

Then, by Lemma 3.2.8, for every $\varepsilon \in (0,\hat{\varepsilon}]$ *there is a* $\rho^*(\varepsilon)$ *such that for every* $\rho \in (0,\rho^*(\varepsilon)]$ *and every* $\tau \in \mathbb{R}$ *we have*

$$\mathcal{D}_{\varepsilon,\rho}^{loc}(\tau) \subset W(\varepsilon).$$

Vice versa there is a neighborhood $V(\varepsilon) \subset X \times Y$ *of* 0 *such that for all* $(x,y) \in \mathcal{X}_\varepsilon(\tau) \cap V(\varepsilon)$ *and all* $(x',y') \in \mathcal{Y}_\varepsilon(\tau) \cap V(\varepsilon)$ *we have*

$$\mu_\varepsilon(t,\tau,x',y',u^h) = \psi(t,\tau,x',y',u^h) = \mathcal{F}(t)\varphi(t,\tau,\mathcal{F}^{-1}(\tau)(x',y'),u^h) \text{ for all } t \leq \tau,$$
$$\mu_\varepsilon(t,\tau,x',y',u^h) = \psi(t,\tau,x',y',u^h) = \mathcal{F}(t)\varphi(t,\tau,\mathcal{F}^{-1}(\tau)(x',y'),u^h) \text{ for all } t \leq \tau.$$

The next proposition gives us an answer to the following question. If, for example, we have a continuous curve $c(t) \in \mathcal{X}_\varepsilon^{loc}(0)$ with $\lim_{t\to\infty} c(t) = 0$, is there a $k \in \mathbb{N}$ and a time $t > 0$ such that we have $c(t) \in \varphi(-2k\Theta, 0, \mathcal{T}_{\varepsilon,\rho}(0))$? The result will be used in the next chapter. The problem here is, that $\mathcal{T}_{\varepsilon,\rho}(\tau)$ is a homeomorphic image of a sphere, which lies on the topological manifold $\mathcal{X}_\varepsilon^{loc}(0)$. To see in general, if a homeomorphic image of a sphere divides a space into two open and connected components is not trivial, which can be seen at the example of the Jordan curve Theorem (cf. for example Engelking and Sieklucki [13], Theorem 4.2.5). But in our case, we can solve this without using algebraic topology.

Proposition 3.2.11 *Let* $\varepsilon \in (0,\hat{\varepsilon}], \rho \in (0,\rho^*(\varepsilon)], \tau \in \mathbb{R}$ *and let* $c : [0,\infty) \to \mathbb{R}^d$ *be a continuous curve with* $c(t) \in \mathcal{X}_{\varepsilon,\rho}^{loc}(\tau)$ *for all* $t \geq 0$ *and* $\lim_{t\to\infty} c(t) = 0$. *Then for every* $K_0 \in \mathbb{N}$ *and for every* $S_0 > 0$ *there is a time* $S \geq S_0$ *and a* $k \in \mathbb{N}$ *with* $k > K_0$ *such that*

$$c(S) \in \varphi(-2k\Theta + \tau, \tau, \mathcal{T}_{\varepsilon,\rho}(\tau), u^h).$$

Proof. For $d = 1$ the statement is true, because $\mathcal{X}_\varepsilon^{loc}(\tau)$ is a continuous curve.

Now let $d \geq 2$. By Lemma 6.2.9, there is a neighborhood $W(\varepsilon) \subset \mathbb{R}^d$ such that for all $p \in \mathcal{X}_\varepsilon^{loc}(\tau) \cap W(\varepsilon)$ we have

$$\varphi(t,\tau,p,u^h) = \mathcal{F}^{-1}(t)\psi(t,\tau,\mathcal{F}(t)p,u^h) = \mathcal{F}^{-1}(t)\mu_\varepsilon(t,\tau,\mathcal{F}(\tau)p,u^h) \text{ for all } t \leq \tau.$$

Since $\lim_{t\to\infty} c(t) = 0$ there is a $t_0 \geq S_0$ such that $c(t) \in W(\varepsilon)$ for all $t > t_0$. Define the continuous function

$$d : [t_0,\infty) \to X \times Y, t \longmapsto \mathcal{H}_\varepsilon(\tau, \mathcal{F}(t)c(t)).$$

Because $c(t) \in W(\varepsilon) \cap \mathcal{X}_\varepsilon^{loc}(\tau)$ for all $t \geq t_0$ it follows that $d(t) \in X \times \{0\}$.

$T_\rho \subset X$ divides the space X into two open pathconnected subsets $V_1 := \{x \in X : \|x\| < \rho\}$ and $V_2 := \{x \in X : \|x\| > \rho\}$. Thus is follows, that for every $k \in \mathbb{N}, i = 1,2$

the sets $\nu_X(-2k\Theta+\tau,\tau,u^h)V_i$ are open and pathconnected subsets of X which do not intersect each other and $0 \in \nu_X(-2k\Theta+\tau,\tau,u^h)V_1$ because $v_X(-2k\Theta+\tau,\tau,u^h)0 = 0$. By relation (6.39) it follows, that there is a $k_0 > K, k_0 \in \mathbb{N}$ such that for all $k > k_0$

$$\begin{aligned}
&\left\|\nu_X(-2k\Theta+\tau,\tau,u^h)x\right\| < \|d(t_0)\| \quad \text{and} \\
&\left\|\nu_X(-2k\Theta+\tau,\tau,u^h)x\right\| < \rho \qquad\qquad \text{for all } x \in V_1 \cup T_\rho.
\end{aligned}$$

This means, that $d(t_0) \in \nu(-2k_0\Theta+\tau,\tau,u^h)V_2$. On the other hand, there is an $a > 0$ such that

$$\left\|\nu_X(-2k\Theta+\tau,\tau,u^h)x\right\| > a \text{ for all } x \in V_2 \cup T_\rho.$$

Because $\lim_{t\to\infty} d(t) = 0$ there is a time $t_1 > t_0$ such that $d(t_1) < a$ and it follows, that $d(t_1) \in \mu_X(-2k\Theta+\tau,\tau,u^h)V_1$.

Now the continuous curve

$$f : [t_0,t_1] \to X, t \mapsto \nu_X(\tau,-2k\Theta+\tau,u^h)d(t)$$

has the property, that $f(t_0) \in V_2$ and $f(t_1) \in V_1$. Thus there exists a time $S \in (t_0,t_1)$ with $f(S) = (x,0) \in T_\rho$. We obtain

$$\begin{aligned}
d(S) &= \nu_X(-2k\Theta+\tau,\tau,u^h)f(S) \\
&= \nu(-2k\Theta+\tau,\tau,u^h)(x,0)^T \in \nu(-2k\Theta+\tau,\tau,u^h)T_\rho.
\end{aligned}$$

Because $(x,0) \in T_\rho$ we have $\mathcal{H}^{-1}(\tau,x,0) \in \mathcal{T}_{\varepsilon,\rho}(\tau)$ and we get $\mu_\varepsilon(t,\tau,\mathcal{H}_\varepsilon^{-1}(\tau,x,0)) \in \mathcal{F}(t)W(\varepsilon)$ for all $t \le \tau$ by assumption. Thus we can lift these solutions of the restricted system to the original system, which means that

$$\begin{aligned}
&\mathcal{F}^{-1}(-2k\Theta+\tau)\circ\mathcal{H}_\varepsilon^{-1}(-2k\Theta+\tau,\nu(-2k\Theta+\tau,\tau,u^h)(x,0)^T) \\
&= \varphi(-2k\Theta+\tau,\tau,\mathcal{F}^{-1}(-2k\Theta+\tau)\circ\mathcal{H}_\varepsilon^{-1}(-2k\Theta+\tau,x,0),u^h).
\end{aligned}$$

By periodicity of $\mathcal{F}^{-1}$ and $\mathcal{H}$ we get

$$\begin{aligned}
c(S) &= \mathcal{F}^{-1}(\tau)\circ\mathcal{H}^{-1}(\tau,d(S)) \\
&= \mathcal{F}^{-1}(-2k\Theta+\tau)\circ\mathcal{H}_\varepsilon^{-1}(-2k\Theta+\tau,\nu(-2k\Theta+\tau,\tau,u^h)(x,0)^T) \\
&= \varphi(-2k\Theta+\tau,\tau,\mathcal{F}^{-1}(-2k\Theta+\tau)\circ\mathcal{H}_\varepsilon^{-1}(-2k\Theta+\tau,x,0),u^h) \\
&= \varphi(-2k\Theta+\tau,\tau,\mathcal{F}^{-1}(\tau)\circ\mathcal{H}^{-1}(\tau,x,0),u^h) \\
&\in \varphi(-2k\Theta+\tau,\tau,\mathcal{T}_\rho(\tau),u^h).
\end{aligned}$$

■

3.3 Steering to the Target

In this section we show, that if we apply the solution map of the hyperbolic system (3.6) to the target and go backwards in time, then the target shrinks in a uniform way towards the singular point 0. Furthermore, if we start at a point close enough to the shrunk set and follow the trajectory in positive time, then we get into a given neighborhood around the target.

We first show this for the restricted system, and then lift this result to the hyperbolic system (3.6).

Lemma 3.3.1 *Let $\varepsilon \in (0, \hat{\varepsilon}], \rho \in (0, \rho^*(\varepsilon)]$. For $\Delta, S > 0$ there is a neighborhood $W \subset X \times Y$ of 0 such that for every open neighborhood $V \subset W$ of 0 we have:*

(a) If $\tau \in \mathbb{R}, (x, y) \in \mathcal{T}_{\varepsilon,\rho}(\tau)$ then there is a $k_0 \in \mathbb{N}$ with $2k_0\Theta > S$ such that for every $k \geq k_0$, $k \in \mathbb{N}$ we have

$$\mu_\varepsilon(-2k\Theta + \tau, \tau, x, y, u^h) \in V. \tag{3.14}$$

(b) If there is $(x', y') \in V$ with $\mathcal{P}_\varepsilon(\tau, x', y') = \mu_\varepsilon(-2k\Theta + \tau, \tau, x, y, u^h)$, then

$$\mu_\varepsilon(t, \tau, x', y', u^h) \in B_\Delta(\mathcal{X}_{\varepsilon,\leq\rho}(t)) \cap V(\varepsilon)$$

for every $t \in [\tau, \tau + 2k\Theta]$ and

$$\mu_\varepsilon(2k\Theta + \tau, \tau, x', y', u^h) \in B_\Delta(\mathcal{T}_{\varepsilon,\rho}(\tau)).$$

$V(\varepsilon)$ is defined as in Notation 3.2.10.

Proof. By Corollary 6.1.21 we have for all $\gamma \in (\beta + KL, \alpha - KL)$ the following estimation

$$\begin{aligned} &\left\| \mu_\varepsilon(t, \tau, x', y', u^h) - \mu_\varepsilon(t, \hat{\mathcal{P}}_\varepsilon(\tau, x', y'), u^h) \right\| \\ &\leq \frac{K(\gamma - \beta)}{\gamma - \beta - KL} e^{\gamma(t-s)} \left\| (x', y') - \mathcal{P}_\varepsilon(\tau, x', y') \right\| \text{ for all } t \geq \tau. \end{aligned}$$

From relation (6.17) and by choosing $\gamma \in (\beta + KL, 0)$ we can find a neighborhood $W \subset V(\varepsilon)$ of 0 such that for all $(x', y') \in W$ we have

$$\left\| \mu_\varepsilon(t, \tau, x', y', u^h) - \mu_\varepsilon(t, \hat{\mathcal{P}}_\varepsilon(\tau, x', y'), u^h) \right\| < \Delta \text{ for all } t \geq \tau. \tag{3.15}$$

According Corollary 6.1.19 for all $\tau \in \mathbb{R}$ we have

$$\left\| \mu_\varepsilon(t, \tau, x, y, u^h) \right\| \leq \frac{K(\alpha - \gamma)}{\alpha - \gamma + KL} \|x\| e^{\gamma(t-\tau)} \text{ for } t \leq \tau$$

for all $\gamma \in (\beta + KL, \alpha - KL)$ and all $(x,y) \in \mathcal{X}_\varepsilon(\tau)$. If we choose $\gamma \in (0, \beta - KL)$, we see that for every neighborhood $V \subset W$ of 0 there is a time $S_0(V) \geq S$ such that for every $\tau \in \mathbb{R}, (x,y) \in \mathcal{T}_{\varepsilon,\rho}(\tau)$ we have

$$\mu_\varepsilon(-t+\tau, \tau, x, y, u^h) \in V \text{ for all } t > S_0(V). \tag{3.16}$$

Thus we have shown assertion (3.14)

Now choose $k_0 \in \mathbb{N}$ with $2k_0\Theta > S_0(V)$. Let $(x,y) \in \mathcal{T}_{\varepsilon,\rho}(\tau)$ and $k > k_0$ with $\mu_\varepsilon(-2k\Theta+\tau, \tau, x, y, u^h) \in V \subset W$ and let $(x', y') \in V$ such that $\mu_\varepsilon(-2k\Theta+\tau, \tau, x, y, u^h) = \mathcal{P}_\varepsilon(\tau, x', y')$. Because of relation (3.15) and $V \subset W$ we get

$$\mu_\varepsilon(t, \tau, x', y', u^h) \in B_\Delta(\mathcal{X}_\varepsilon(t))$$

for all $t \geq \tau$. With

$$\begin{aligned}
\mu_\varepsilon(t, \hat{\mathcal{P}}_\varepsilon(\tau, x', y'), u^h) &= \mu_\varepsilon(t, \tau, \mathcal{P}_\varepsilon(\tau, x', y'), u^h(-2k\Theta + \cdot)) \\
&= \mu_\varepsilon(t - 2k\Theta, \tau - 2k\Theta, \mathcal{P}_\varepsilon(\tau, x', y'), u^h) \\
&= \mu_\varepsilon(t - 2k\Theta, \tau - 2k\Theta, \mu_\varepsilon(\tau - 2k\Theta, \tau, x, y, u^h), u^h) \\
&= \mu_\varepsilon(t - 2k\Theta, \tau, x, y, u^h)
\end{aligned}$$

it follows that $\mu_\varepsilon(t, \hat{\mathcal{P}}_\varepsilon(\tau, x', y'), u^h) \in V(\varepsilon)$ for all $t \in [\tau, \tau + 2k\Theta]$. Finally from

$$\mu_\varepsilon(\tau + 2k\Theta, \hat{\mathcal{P}}_\varepsilon(\tau, x', y'), u^h) = \mu_\varepsilon(\tau, \tau, x, y, u^h) = (x, y)$$

we get

$$\left\| \mu_\varepsilon(2k\Theta + \tau, \tau, x', y', u^h) - (x, y) \right\| \leq \Delta$$

which means, that $\mu_\varepsilon(\tau + 2k\Theta, \tau, x', y', u^h) \in B_\Delta(\mathcal{T}_{\varepsilon,\rho}(\tau))$. ■

Before proving the main result in this section, we need the following technical lemma.

Lemma 3.3.2 *For every $\Delta > 0$ there exists a $\tilde{\Delta} > 0$ with*

$$B_{\tilde{\Delta}}(\mathcal{X}_\varepsilon(\tau)) \subset \mathcal{F}(\tau) B_\Delta(\mathcal{X}_\varepsilon^{loc}(\tau)) \textit{ for all } \tau \in \mathbb{R}.$$

Proof. Note that for $(x,y) \in \mathcal{X}_\varepsilon(\tau)$ we have

$$\mathcal{F}^{-1}(\tau)(x,y) \in \mathcal{X}_\varepsilon^{loc}(\tau).$$

Then with $\tilde{\Delta} < \frac{\Delta}{\|\mathcal{F}^{-1}\|}$ it follows for $(x,y) \in \mathcal{X}_\varepsilon(\tau)$ and $(x', y') \in B_{\tilde{\Delta}}(x, y)$

$$\left\| \mathcal{F}^{-1}(\tau)(x,y) - \mathcal{F}^{-1}(\tau)(x', y') \right\| \leq \left\| \mathcal{F}^{-1} \right\| \left\| (x,y) - (x', y') \right\| \leq \Delta.$$

■

After proving the result for the restricted system, we now consider the hyperbolic system (3.6).

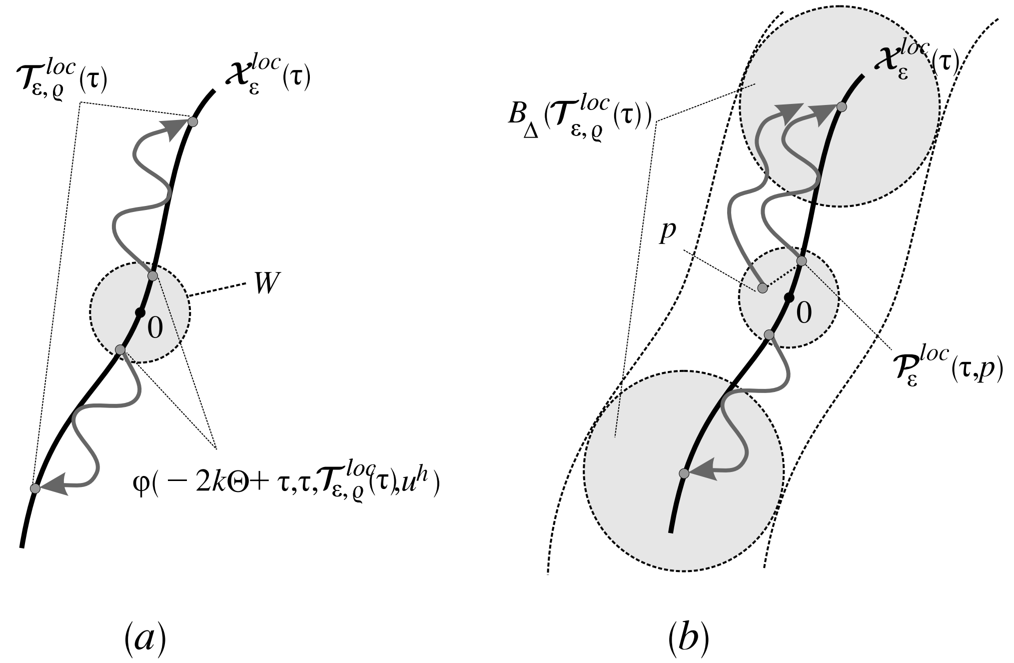

Figure 3.3: Local behavior of the hyperbolic system.

Proposition 3.3.3 *Let $\varepsilon \in (0,\hat{\varepsilon}], \rho \in (0,\rho^*(\varepsilon)]$. For $\Delta, T > 0$ there is a neighborhood $W := W(\Delta,T,\varepsilon,\rho) \subset \mathbb{R}^d$ of 0 such that*

(a) For every $\tau \in \mathbb{R}, q \in \mathcal{T}^{loc}_{\varepsilon,\rho}(\tau)$ there is a $k_0 \in \mathbb{N}$ with $2k_0\Theta > T$ such that for every $k \geq k_0$, $k \in \mathbb{N}$ we have

$$\varphi(-2k\Theta+\tau,\tau,q,u^h) \in W.$$

(b) If $p \in W$ and $\mathcal{P}^{loc}_{\varepsilon}(\tau,p) = \varphi(-2k\Theta+\tau,\tau,q,u^h)$, then

$$\varphi(t,\tau,p,u^h) \in B_\Delta(\mathcal{X}^{loc}_{\varepsilon}(t))$$

for every $t \in [\tau,\tau+2k\Theta]$ and

$$\varphi(2k\Theta+\tau,\tau,p,u^h) \in B_\Delta(\mathcal{T}^{loc}_{\varepsilon,\rho}(\tau)).$$

For an illustration of this result in the case $\dim X = 1$ consider Figure 3.3.

Proof. According to Lemma 3.3.2 there is a $\tilde{\Delta} > 0$ with

$$B_{\tilde{\Delta}}(\mathcal{X}_\varepsilon(\tau)) \subset \mathcal{F}(\tau)B_\Delta(\mathcal{X}^{loc}_\varepsilon(\tau)) \text{ for all } \tau \in \mathbb{R}. \tag{3.17}$$

By Lemma 3.3.1 there exists a neighborhood $\tilde{W} \subset X \times Y$ of 0 such that for every open neighborhood $\tilde{V} \subset X \times Y$ of 0 with $\tilde{V} \subset \tilde{W}$ we have: For every $(x,y) \in \mathcal{T}_{\varepsilon,\rho}(\tau)$ there is a $k_0 \in \mathbb{N}$ with $2k_0\Theta > T$ such that for every $k \geq k_0$, $k \in \mathbb{N}$ we have

$$\mu_\varepsilon(-2k\Theta+\tau,\tau,x,y,u^h) \in \tilde{V}.$$

Furthermore, if $(x',y') \in \tilde{V}$ and $\mathcal{P}_\varepsilon(\tau,x',y') = (x,y)$, then

$$\mu_\varepsilon(t,\tau,x',y',u^h) \in B_{\tilde{\Delta}}(\mathcal{X}_\varepsilon(t)) \cap V(\varepsilon) \text{ for every } t \in [\tau,\tau+2k\Theta],$$
$$\mu_\varepsilon(2k\Theta+\tau,\tau,x',y',u^h) \in B_{\tilde{\Delta}}(\mathcal{T}_{\varepsilon,\rho}(\tau)).$$

Now for such a $\tilde{V}$ there is a neighborhood $W \subset W(\varepsilon) \subset \mathbb{R}^d$ of 0 with

$$W \subset \cap_{t\in\mathbb{R}} \mathcal{F}^{-1}(t)\tilde{V} \subset \mathbb{R}^d$$

and there is a neighborhood $\tilde{V}_0 \subset \tilde{W}$ of 0 such that

$$\tilde{V}_0 \subset \cap_{t\in\mathbb{R}} \mathcal{F}(t)W$$

Let $q \in \mathcal{T}_\rho^{loc}(\tau)$ and define $(x,y) := \mathcal{F}(\tau)q \in \mathcal{T}_\rho(\tau)$. Then there is a $k_0 \in \mathbb{N}$ with $2k_0\Theta > T$ such that for every $k \geq k_0$, $k \in \mathbb{N}$ we have $\mu_\varepsilon(-2k\Theta+\tau,\tau,x,y,u^h) \in \tilde{V}_0$. Because of $\mu_\varepsilon(t,\tau,x,y,u^h) \in V(\varepsilon)$ for every $t \leq \tau$, we get $\varphi(t,\tau,q,u^h) = \mathcal{F}^{-1}(t)\mu_\varepsilon(t,\tau,x,y,u^h)$ for every $t \leq \tau$. Hence

$$\begin{aligned}\varphi(-2k\Theta+\tau,\tau,q,u^h) &= \mathcal{F}^{-1}(-2k\Theta+\tau)\mu_\varepsilon(-2k\Theta+\tau,\tau,x,y,u^h)\\ &\in \mathcal{F}^{-1}(\tau)\tilde{V}_0 \subset W.\end{aligned}$$

For $p \in W$ with $\mathcal{P}_\varepsilon^{loc}(\tau,p) = \varphi(-2k\Theta+\tau,\tau,q,u^h)$ we define $(x',y') := \mathcal{F}(\tau)p \in \tilde{V}$ with $\mathcal{P}_\varepsilon(\tau,x',y') = \mu_\varepsilon(-2k\Theta+\tau,\tau,x,y,u^h)$. Because $\mu_\varepsilon(t,\tau,x',y',u^h) \in V(\varepsilon)$ we get $\varphi(t,\tau,q,u^h) = \mathcal{F}^{-1}(t)\mu_\varepsilon(t,\tau,x',y',u^h)$ for every $t \in [\tau,\tau+2k\Theta]$. With (3.17) we obtain

$$\begin{aligned}\varphi(t,\tau,p,u^h) &= \mathcal{F}^{-1}(t)\mu_\varepsilon(t,\tau,x',y',u^h) \in \mathcal{F}^{-1}(t)(B_{\tilde{\Delta}}(\mathcal{X}_\varepsilon(t))\\ &\subset \mathcal{F}^{-1}(t)\mathcal{F}(t)B_\Delta(\mathcal{X}_\varepsilon^{loc}(t)) = B_\Delta(\mathcal{X}_\varepsilon^{loc}(t)).\end{aligned}$$

for every $t \in [\tau,\tau+2k\Theta]$ and finally

$$\begin{aligned}\varphi(2k\Theta+\tau,\tau,p,u^h) &= \mathcal{F}^{-1}(2k\Theta+\tau)\mu_\varepsilon(2k\Theta+\tau,\tau,x,y,u^h))\\ &\in \mathcal{F}^{-1}(\tau)B_{\tilde{\Delta}}(\mathcal{T}_{\varepsilon,\rho}(\tau))) \subset B_\Delta(\mathcal{T}_{\varepsilon,\rho}^{loc}(\tau)).\end{aligned}$$

■

3.4 Adjusting the neighborhoods

Before we start with the construction of the control function as explained in Section 3.1, we have to fix the neighborhoods which result from the linearization of the nonlinear control system. Only in these neighborhoods we are able to characterize the nonlinear system.

First we choose an $\varepsilon \in (0,\hat{\varepsilon}]$, which means that there is a neighborhood $W(\varepsilon) \subset \mathbb{R}^d$ of 0 such that for all $p \in \mathcal{X}_\varepsilon^{loc}(\tau) \cap W(\varepsilon)$ and all $q \in \mathcal{Y}_\varepsilon^{loc}(\tau) \cap W(\varepsilon)$ we have

$$\begin{aligned}\varphi(t,\tau,p,u^h) = \mathcal{F}^{-1}(t)\psi(t,\tau,\mathcal{F}(\tau)p,u^h) = \mathcal{F}^{-1}(t)\mu_\varepsilon(t,\tau,\mathcal{F}(\tau)p,u^h) \text{ for all } t \leq \tau,\\ \varphi(t,\tau,q,u^h) = \mathcal{F}^{-1}(t)\psi(t,\tau,\mathcal{F}(\tau)q,u^h) = \mathcal{F}^{-1}(t)\mu_\varepsilon(t,\tau,\mathcal{F}(\tau)q,u^h) \text{ for all } t \geq \tau,\end{aligned} \tag{3.18}$$

according to Lemma 6.2.9. Note that the system

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i^s(t) f_i(x) \tag{3.19}$$

(which we call the *stable* system) is locally asymptotically stable (cf. Lemma 6.2.10). Thus we can choose a neighborhood $V^s \subset \mathbb{R}^d$ of 0 such that

$$V^s \subset W(\varepsilon)$$

and an open neighborhood $W^s \subset \mathbb{R}^d$ of 0, such that for all $p \in W^s$ we have

$$\varphi(t, 0, p, u^s) \in V^s \text{ for all } t \geq 0 \text{ and } \lim_{t\to\infty} \varphi(t, 0, p, u^s) = 0. \tag{3.20}$$

By Notation 3.2.10 there is a $\hat{\rho} := \hat{\rho}(\varepsilon, W^s) \in (0, \rho^*(\varepsilon))$ such that

$$\mathcal{D}^{loc}_{\varepsilon,\rho}(t) \subset W^s \text{ for all } \rho \in (0, \hat{\rho}) \text{ and all } t \in \mathbb{R}. \tag{3.21}$$

Thus for all $\rho \in (0, \hat{\rho})$ we obtain

$$\mathcal{T}^{loc}_{\varepsilon,\rho}(t) \subset W^s \subset V^s \subset W(\varepsilon) \text{ for all } t \in \mathbb{R}. \tag{3.22}$$

Furthermore, we assume that the nonlinear control system is locally accessible on $\mathbb{R}^d \setminus \{0\}$.

3.5 The Construction of the Control Function

Now we accomplish the construction of the control function. As mentioned in Section 3.1, the idea is to switch between u^s and u^h. We do this in such a way, that the corresponding trajectory gets closer and closer to the target, if we steer the system away form the singular point 0. On the other hand, we steer the trajectory each time closer to the singular point 0, if we apply u^s.

We fix $\tau \in \mathbb{R}, \varepsilon \in (0, \hat{\varepsilon}], W^s \subset \mathbb{R}^d$ and $\rho \in (0, \hat{\rho})$.

First we choose a sequence $(\delta_i)_{i\in\mathbb{N}} \subset \mathbb{R}^+$ with $\lim_{i\to\infty} \delta_i = 0$ and

$$B_{\delta_i}(\mathcal{D}^{loc}_{\varepsilon,\rho}(t)) \subset W^s \text{ for all } i \in \mathbb{N}, t \in \mathbb{R},$$

which is possible because of relation (3.21).

According to Proposition 3.3.3 there exists a sequence $(\sigma_i)_{i\in\mathbb{N}} \subset \mathbb{R}^+$ with $\lim_{i\to\infty} \sigma_i = 0$, such that we have: For every $\tau \in \mathbb{R}, q \in \mathcal{T}^{loc}_{\varepsilon,\rho}(\tau)$ there is a $K_0 \in \mathbb{N}$ with $2K_0\Theta > 2^i$ such that for every $k \geq K_0$, $k \in \mathbb{N}$ we have $\varphi(-2k\Theta + \tau, \tau, q, u^h) \in B_{\sigma_i}(0)$. Furthermore, if $p \in B_{\sigma_i}(0)$ and $\mathcal{P}^{loc}_{\varepsilon}(\tau, p) = \varphi(-2k\Theta + \tau, \tau, q, u^h)$, then

$$\varphi(t, \tau, p, u^h) \in B_{\delta_{i+1}}(\mathcal{X}^{loc}_{\varepsilon}(t)) \text{ for every } t \in [\tau, \tau + 2k\Theta] \text{ and}$$
$$\varphi(2k\Theta + \tau, \tau, p, u^h) \in B_{\delta_{i+1}}(\mathcal{T}^{loc}_{\varepsilon,\rho}(\tau)).$$

For the construction of our trajectory we

$$\textit{start at an arbitrary point } p := p_{-1} \in \mathcal{T}^{loc}_{\varepsilon,\rho}(\tau).$$

Now we define the control function recursively, by counting the variable i. We start with $i = 0$.

Step 0:
• Since $p \in W^s$ there is a time $\Delta\tilde{T}_0$ with

$$\varphi(t, 0, p, u^s) \in B_{\rho_0}(0) \text{ for all } t \geq \Delta\tilde{T}_0.$$

Compare Figure 3.4 (a), where we set $i = 0$.▲

Now there are two cases:

• *First case: For all $t \geq \Delta\tilde{T}_0$ we have $P^{loc}_\varepsilon(\tau, \varphi(t, 0, p, u^s)) \neq 0$.* For this case consider Figure 3.4 (b). We obtain

$$\varphi(t, 0, p, u^s) \notin \mathcal{Y}^{loc}_\varepsilon(\tau) \text{ for all } t \geq \Delta\tilde{T}_0,$$

because $\mathcal{P}^{loc}_\varepsilon(\tau, q) = 0$ iff $q \in \mathcal{Y}^{loc}_\varepsilon(\tau)$.

$\mathcal{P}^{loc}_\varepsilon$ is continuous and $\mathcal{P}^{loc}_\varepsilon(\tau, 0) = 0$. Thus, by $\lim_{t\to\infty} \varphi(t, 0, p, u^s) = 0$ it follows that $\lim_{t\to\infty} \mathcal{P}^{loc}_\varepsilon(\tau, \varphi(t, 0, p, u^s)) = 0$. Note, that by relation (3.22) we can apply Proposition 3.2.11 and we get, that there is $k_0 \in \mathbb{N}$ with $2k_0\Theta > 2^0$ and $\Delta T_0 > \Delta\tilde{T}_0, q_0 \in \mathcal{T}^{loc}_{\varepsilon,\rho}(0)$ with

$$\mathcal{P}^{loc}_\varepsilon(\tau, \varphi(\Delta T_0, 0, p, u^s)) = \varphi(-2k_0\Theta + \tau, \tau, q_0, u^h).$$

Define

$$\begin{aligned} p_0 := p_1 &:= \varphi(\Delta T_0, 0, p, u^s), \\ \Delta T_1 &:= 0 \text{ and} \\ u_0 &\in \mathcal{U} \text{ arbitrary.} \end{aligned}$$

This completes the first case.▲

• *Second case: There is a $\Delta T_0 \geq \Delta\tilde{T}_0$ such that $P^{loc}_{\varepsilon,\rho}(\tau, \varphi(\Delta T_0, 0, p, u^s)) = 0$.*

For this case consider Figure 3.4 (c). Then we define

$$p_0 := \varphi(\Delta T_0, 0, p, u^s)$$

with $p_0 \in \mathcal{Y}^{loc}_\varepsilon(\tau)$. Since we assumed local accessibility on $\mathbb{R}^d \setminus \{0\}$, there is a control function $u_0 \in \mathcal{U}$ and a time $0 < \Delta\tilde{T}_1 < 1$, such that

$$\begin{aligned} \varphi(t, 0, p_0, u_0) &\in B_{\sigma_0}(0) \text{ for all } t \in [0, \Delta\tilde{T}_1] \text{ and} \\ \varphi(\Delta\tilde{T}_1, 0, p_0, u_0) &\notin \mathcal{Y}^{loc}_\varepsilon(\tau). \end{aligned}$$

Then there is an interval $[a_0, b_0] \subset [0, \Delta\tilde{T}_1]$ with

$$\varphi(a_0, 0, p_1, u_0) \in \mathcal{Y}_\varepsilon^{loc}(\tau) \text{ and}$$
$$\varphi(t, 0, p_1, u_0) \notin \mathcal{Y}_\varepsilon^{loc}(\tau) \text{ for all } t \in (a_0, b_0].$$

Because $\varphi(a_0, 0, p_1, u_0) \in \mathcal{Y}_\varepsilon^{loc}(\tau)$ we have

$$\mathcal{P}_\varepsilon^{loc}(\tau, \varphi(a_0, \tau, p_1, u_0)) = 0,$$

and because $\varphi(t, 0, p_1, u_0) \notin \mathcal{Y}_\varepsilon^{loc}(\tau)$ we get

$$\mathcal{P}_\varepsilon^{loc}(\tau, \varphi(t, 0, p_1, u_0)) \in \mathcal{X}_\varepsilon^{loc}(\tau) \setminus \{0\} \text{ for all } t \in (a_0, b_0].$$

We apply Proposition 3.2.11 which yields a $k_0 \in \mathbb{N}$ such that $2k_0\Theta > 2^0$ and $q_0 \in \mathcal{T}_{\varepsilon,\rho}^{loc}(\tau)$, $\Delta T_1 \in (a_0, b_0]$ with

$$\mathcal{P}_\varepsilon^{loc}(\tau, \varphi(\Delta T_1, 0, p_0, u_0)) = \varphi(-2k_0\Theta + \tau, \tau, q_0, u^h).$$

Define

$$p_1 := \varphi(\Delta T_1, 0, p_0, u_0),$$

which completes the second case.▲

- Now in both cases, we stopped with a point $p_1 \in B_{\sigma_0}(0)$ and $q_0 \in \mathcal{T}_{\varepsilon,\rho}^{loc}(\tau)$ with

$$\mathcal{P}_\varepsilon^{loc}(\tau, p_1) = \varphi(-2k_0\Theta + \tau, \tau, q_0, u^h).$$

Consider Figure 3.4 (d). Define

$$\Delta T_2 := 2k_0\Theta \text{ and}$$
$$p_2 := \varphi(\Delta T_2 + \tau, \tau, p_1, u^h) = \varphi(\Delta T_2, 0, p_1, u^h(\tau + \cdot))$$

By construction we have

$$p_2 \in B_{\delta_1}(\mathcal{T}_{\varepsilon,\rho}^{loc}(\tau)).$$

This is the end of Step 0.▲

We define the times ΔT_i, the points p_i and the controls $u_i \in \mathcal{U}$ for $i = 1, 2, \ldots$ recursively.

Step i:

Consider Figure 3.1.

- By construction, we have $p_{3i-1} \in B_{\delta_i}(\mathcal{T}_{\varepsilon,\rho}^{loc}(\tau))$. Then there is a time $\Delta\tilde{T}_{3i} > 2^i$ with

$$\varphi(t, 0, p_{3i-1}, u^s) \in B_{\sigma_i}(0) \text{ for all } t \geq \Delta\tilde{T}_{3i}.$$

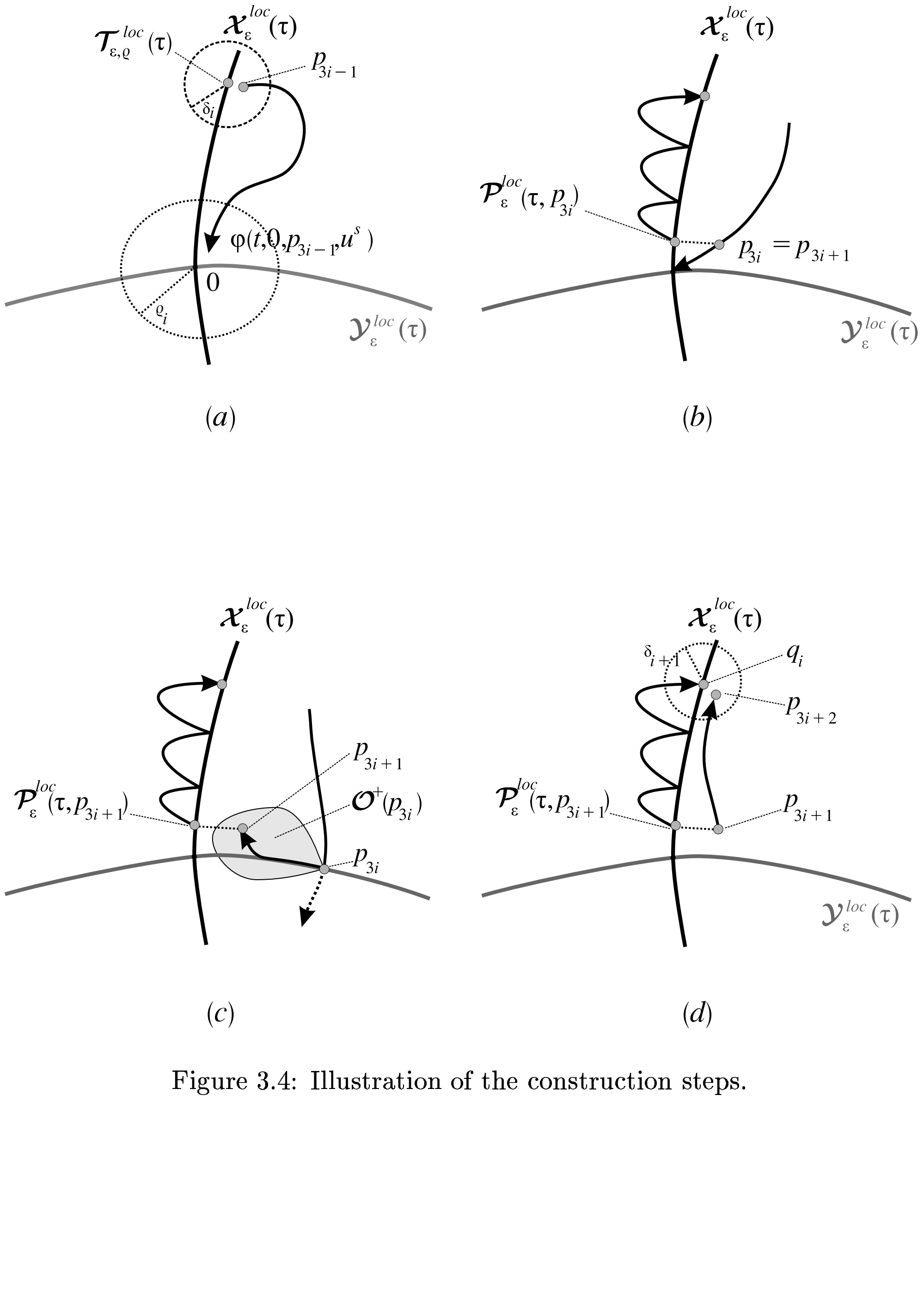

Figure 3.4: Illustration of the construction steps.

Consider Figure 3.4 (a).▲

Now there are again two cases:

• *First case: For all* $t \geq \Delta\tilde{T}_{3i}$ *we have* $\mathcal{P}_\varepsilon^{loc}(\tau, \varphi(t, 0, p_{3i-1}, u^s)) \neq 0$.

For this case consider Figure 3.4 (b). Then we have $\varphi(t, 0, p_{3i-1}, u^s) \notin \mathcal{Y}_\varepsilon^{loc}(\tau)$ for all $t \geq \Delta\tilde{T}_{3i}$, and $\lim_{t\to\infty} \mathcal{P}_\varepsilon^{loc}(\tau, \varphi(t, 0, p_{3i-1}, u^s)) = 0$. Thus by Proposition 3.2.11 it follows, that there is a $k_i \in \mathbb{N}$ with $2k_i\Theta > 2^i$ and $\Delta T_{3i} > \Delta\tilde{T}_{3i}, q_i \in \mathcal{T}_{\varepsilon,\rho}^{loc}(\tau)$ such that

$$\mathcal{P}_\varepsilon^{loc}(\tau, \varphi(\Delta T_{3i}, 0, p_{3i-1}, u^s)) = \varphi(-2k_i\Theta + \tau, \tau, q_i, u^h).$$

Define

$$\begin{aligned} p_{3i} &:= p_{3i+1} := \varphi(\Delta T_{3i}, 0, p_{3i-1}, u^s), \\ \Delta T_{3i+1} &:= 0, \\ &\text{and } u_i \in \mathcal{U} \text{ arbitrarily.} \end{aligned}$$

This completes the first case.▲

• *Second case: There is a* $\Delta T_{3i} \geq \Delta\tilde{T}_{3i}$ *such that* $P_\varepsilon^{loc}(\tau, \varphi(\Delta T_{3i}, 0, p_{3i-1}, u^s)) = 0$.

For this case consider Figure 3.4 (c).Then we define

$$p_{3i} := \varphi(\Delta T_{3i}, 0, p_{3i-1}, u^s)$$

with $p_{3i} \in \mathcal{Y}_\varepsilon^{loc}(\tau)$. Since we assumed local accessibility on $\mathbb{R}^d \setminus \{0\}$ there is a control function $u_i \in \mathcal{U}$ and a time $0 < \Delta\tilde{T}_{3i+1} < 2^{-i}$, such that

$$\begin{aligned} \varphi(t, 0, p_{3i}, u_i) &\in B_{\sigma_i}(0) \text{ for all } t \in [0, \Delta\tilde{T}_{3i+1}] \text{ and} \\ \varphi(\Delta\tilde{T}_{3i+1}, 0, p_{3i}, u_i) &\notin \mathcal{Y}_\varepsilon^{loc}(\tau). \end{aligned}$$

Then there is an interval $[a_i, b_i] \subset [0, \Delta\tilde{T}_{3i+1}]$ with

$$\begin{aligned} &\varphi(a_i, 0, p_{3i}, u_i) \in \mathcal{Y}_\varepsilon^{loc}(\tau) \text{ and} \\ &\varphi(t, 0, p_{3i}, u_i) \notin \mathcal{Y}_\varepsilon^{loc}(\tau) \text{ for all } t \in (a_i, b_i]. \end{aligned}$$

Because $\varphi(a_i, 0, p_{3i}, u_i) \in \mathcal{Y}_\varepsilon^{loc}(\tau)$ we have

$$\mathcal{P}_\varepsilon^{loc}(\tau, \varphi(a_i, 0, p_{3i}, u_i)) = 0,$$

and because $\varphi(t, 0, p_{3i}, u_i) \notin \mathcal{Y}_\varepsilon^{loc}(\tau)$ we get for all $t \in (a_i, b_i]$

$$\mathcal{P}_\varepsilon^{loc}(\tau, \varphi(t, 0, p_{3i}, u_i)) \in \mathcal{X}_\varepsilon^{loc}(\tau) \setminus \{0\}.$$

Thus, according to Proposition 3.2.11, there is a $k_i \in \mathbb{N}$ with $2k_i\Theta > 2^i$ and $q_i \in \mathcal{T}_{\varepsilon,\rho}^{loc}(\tau), \Delta T_{3i+1} \in (a_i, b_i]$ such that

$$\mathcal{P}_\varepsilon^{loc}(\tau, \varphi(\Delta T_{3i+1}, 0, p_{3i}, u_i)) = \varphi(-2k_i\Theta + \tau, \tau, q_i, u^h).$$

Define

$$p_{3i+1} := \varphi(\Delta T_{3i+1}, p_{3i}, u_i).$$

This completes the second case.▲

• Now in both cases, we stopped with a point $p_{3i+1} \in B_{\rho_i}(0) \cap \mathcal{X}_\varepsilon^{loc}(\tau)$ and a $q_i \in \mathcal{T}_{\varepsilon,\rho}^{loc}(\tau)$ such that

$$\mathcal{P}_\varepsilon^{loc}(\tau, p_{3i+1}) = \varphi(-2k_i\Theta + \tau, \tau, q_i, u^h).$$

Consider Figure 3.4 (d). Define

$$\begin{aligned} \Delta T_{3i+2} &:= 2k_i\Theta \text{ and} \\ p_{3i+2} &:= \varphi(\Delta T_{3i+2} + \tau, \tau, p_{3i+1}, u^h) = \varphi(\Delta T_{3i+2}, 0, p_{3i+1}, u^h(\tau + \cdot)). \end{aligned}$$

By construction, $p_{3i+2} \in B_{\delta_{i+1}}(\mathcal{T}_{\varepsilon,\rho}^{loc}(\Delta T_{3i+2} + \tau)) = B_{\delta_{i+1}}(\mathcal{T}_{\varepsilon,\rho}^{loc}(\tau))$. This is the end of step i, and the same procedure starts again with Step $i + 1$.▲

We define

$$T_i := \sum_{k=0}^{i} \Delta T_k$$

and the function $u : \mathbb{R} \to U$ by

$$u(t) := \begin{cases} 0 & \text{for } t < 0, \\ u^s(t - T_{3i-1}) & \text{for } t \in [T_{3i-1}, T_{3i}), \\ u_i(t - T_{3i}) & \text{for } t \in [T_{3i}, T_{3i+1}), \\ u^h(t + \tau - T_{3i+1}) & \text{for } t \in [T_{3i+1}, T_{3i+2}), \end{cases}$$

for $i = 0, 1, 2, \ldots$ and $T_{-1} := 0$.

Remark 3.5.1 *The function u depends on the chosen constants ε, ρ. In the construction process above, we started with a fixed time $\tau \in \mathbb{R}$, and have chosen a point $p \in \mathcal{T}_{\varepsilon,\rho}^{loc}(\tau)$. Therefore all the times ΔT_i and the control functions u_i depend on τ, p, ε and ρ. For indicating this, we will denote u by $u_{\varepsilon,\rho,\tau,p}$. But $u_{\varepsilon,\rho,\tau,p}$ is* not *uniquely determined by ε, ρ, τ and p, because we may choose the ΔT_{3i} as large as we want.*

Remark 3.5.2 *By local uniqueness of the unstable and stable fibre bundles (see Proposition 6.2.11) it follows, that for a given $\varepsilon \in (0, \hat{\varepsilon}]$ there is a neighborhood $W \subset \mathbb{R}$ such that for every $\varepsilon' \in [\varepsilon, \hat{\varepsilon}]$ we have*

$$\begin{aligned} \mathcal{X}_\varepsilon^{loo}(t) \cap W &= \mathcal{X}_{\varepsilon'}^{loc}(t) \cap W \\ \mathcal{Y}_\varepsilon^{loo}(t) \cap W &= \mathcal{Y}_{\varepsilon'}^{loc}(t) \cap W. \end{aligned}$$

Thus if we chose $V^s \subset W(\varepsilon) \cap W$ and $W^s, \hat{\rho}, \rho$ as in Section 3.4, the construction of the control function $u_{\varepsilon,\rho,\tau,p}$ stays valid with the same times ΔT_i if we replace $\mathcal{X}_\varepsilon^{loc}$ by $\mathcal{X}_{\varepsilon'}^{loc}$ and $\mathcal{Y}_\varepsilon^{loc}$ by $\mathcal{Y}_{\varepsilon'}^{loc}$. Thus we may conclude

$$u_{\varepsilon,\rho,\tau,p} = u_{\varepsilon',\rho,\tau,p}.$$

3.6 The Limit Set

We want to apply Proposition 1.1.21 to get the existence of a control set with nonvoid interior. The proposition demands the existence of a compact subset of $\mathcal{U} \times \mathbb{R}^d$ which has nonvoid intersection with the ω-limit set $\omega(u, x)$, on which two inner pair conditions are satisfied.

If we take a pair $(u^*, p^*) \in \omega(u, p)$, then for every $-\infty < T_1 \leq T_2 < \infty$ the set

$$\{\Phi_t(u^*, p^*) : t \in [T_1, T_2]\}$$

is a compact subset of $\omega(u, p)$. This will be used in the next section to prove the Existence Theorem 3.7.1 for control sets near the singular point $x^* = 0$.

Here we show that the ω-limit set has nonvoid intersection with the target $\mathcal{T}^{loc}_{\varepsilon,\rho}(\tau)$.

Theorem 3.6.1 *Let $\tau \in \mathbb{R}, p \in \mathcal{T}^{loc}_{\varepsilon,\rho}(\tau)$ and let $u := u_{\varepsilon,\rho,\tau,p} \in \mathcal{U}$ be constructed as in Section 3.5. Then there is a $p^* \in \mathcal{T}^{loc}_{\varepsilon,\rho}(\tau) \subset \mathcal{X}^{loc}_{\varepsilon}(\tau)$ such that*

$$\{\Phi_\tau(u^*(\cdot, \tau), p^*) : \tau \in \mathbb{R}\} \subset \omega(u, p)$$

where $u^* : \mathbb{R} \times \mathbb{R} \to U$ is defined by

$$u^*(t, \tau) := \begin{cases} u^s(t) & \text{for } t \geq 0, \\ u^h(t + \tau) & \text{for } t < 0. \end{cases}$$

Proof. Fix $\tau \in \mathbb{R}$ and for abbreviation write $u^*(\cdot) := u^*(\cdot, \tau)$. First we show, that

$$\lim_{k \to \infty} \theta_{T_{3k-1}} u = u^*.$$

For that purpose, remember that $\mathcal{U}$ is supplied with the weak*-topology of $L_\infty(\mathbb{R}, \mathbb{R}^m)$. Let $W \subset \mathcal{U}$ be a neighborhood of u^*. Then there exists a $\sigma > 0$ and $g_1, \ldots, g_n \in L^1(\mathbb{R}, \mathbb{R}^m)$ such that

$$\left\{ v \in L^\infty(\mathbb{R}, \mathbb{R}^m) : \begin{array}{l} \left|\int_{\mathbb{R}} \langle u^*(t) - v(t), g_j(t) \rangle \, dt\right| < \sigma \\ \text{for } j = 1, \ldots, n, \text{ and } v(t) \in U, \forall t \in \mathbb{R} \end{array} \right\} \subset W, \tag{3.23}$$

because the sets of this form are a subbasis of the weak*-topology (cf. for example Dunford and Schwartz [12]). We show, that for all $g \in L^1(\mathbb{R}, \mathbb{R}^m)$ and all $\sigma > 0$ there is a $N \in \mathbb{N}$ with

$$\left| \int_{\mathbb{R}} \langle u^*(t) - \theta_{T_{3k}} u(t), g(t) \rangle \, dt \right| < \sigma \text{ for all } k > N.$$

Then it follows that $\theta_{T_{3k}} u$ is an element of the set on the left hand side of (3.23) and hence $\theta_{T_{3k}} u \in W$ for all $k \in \mathbb{N}$ big enough.

So let $g \in L^1(\mathbb{R}, \mathbb{R}^m)$ and $\sigma > 0$. Then there exists a time $T > 0$ with

$$\int_{\mathbb{R} \setminus [-T, T]} |g(t)| \, dt < \frac{\sigma}{2 \text{diam} U}.$$

Furthermore, because $\Delta T_{3i} > 2^k$ and $\Delta T_{3i-1} > 2^k$ for all $k \in \mathbb{N}$ by construction of the control function $u \in \mathcal{U}$, there exists a $N \in \mathbb{N}$ with

$$\Delta T_{3k-1} > T \text{ and}$$
$$\Delta T_{3k} > T \text{ for all } k > N.$$

This guarantees, that

$$\begin{aligned} u(t + T_{3k-1}) &= u^s(t) && \text{for all } t \in [0,T] \text{ and} \\ u(t + T_{3k-1}) &= u^h(t+\tau) && \text{for all } t \in [-T,0]. \end{aligned}$$

Then we get

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \langle u(t+T_{3k-1}) - u^*(t), g(t)\rangle \, dt\right| \\
&\leq \left|\int_{\mathbb{R}^-} \langle u(t+T_{3k-1}) - u^*(t), g(t)\rangle \, dt\right| + \left|\int_{\mathbb{R}^+} \langle u(t+T_{3k-1}) - u^*(t), g(t)\rangle \, dt\right| \\
&\leq \left|\int_{\mathbb{R}^-\setminus[-T,0]} \langle u(t+T_{3k-1}) - u^*(t), g(t)\rangle \, dt\right| + \left|\int_{\mathbb{R}^+\setminus[0,T]} \langle u(t+T_{3k-1}) - u^*(t), g(t)\rangle \, dt\right| \\
&+ \left|\int_{[-T,0]} \langle u(t+T_{3k-1}) - u^*(t), g(t)\rangle \, dt\right| + \left|\int_{[0,T]} \langle u(t+T_{3k-1}) - u^*(t), g(t)\rangle \, dt\right| \\
&\leq \sigma + \left|\int_{[-T,0]} \left\langle u^h(t+\tau) - u^*(t), g(t)\right\rangle dt\right| + \left|\int_{[0,T]} \langle u^s(t) - u^*(t), g(t)\rangle \, dt\right| \\
&= \sigma.
\end{aligned}$$

By construction we have for all $k \in \mathbb{N}$

$$\varphi(T_{3k-1}, 0, p, u) \in B_{\delta_{k+1}}(\mathcal{T}^{loc}_{\varepsilon,\rho}(\tau)).$$

Since $\mathcal{T}^{loc}_{\varepsilon,\rho}(\tau)$ is compact, there exists a subsequence $(T_{3k_l})_{l\in\mathbb{N}}$ with

$$p^* := \lim_{l\to\infty} \varphi(T_{3k_l-1}, 0, p, u) \in \mathcal{T}^{loc}_{\varepsilon,\rho}(\tau).$$

Thus it follows, that $(u^*, p^*) \in \omega(u,p)$. Because $\omega(u,p)$ is invariant (cf. Corollary 1.1.16), the assertion follows. ■

Remark 3.6.2 *The point p^* is* not *uniquely defined. As one sees in the proof, p^* is only given by some convergent subsequence of $(\varphi(T_{3k-1}, 0, p, u))_{k\in\mathbb{N}}$.*

3.7 The Existence Theorem

After having made the construction of the control set $u \in \mathcal{U}$ in Section 3.5 and the characterization of the ω-limit set, we finally use this to prove the following existence theorem.

Theorem 3.7.1 *Consider the nonlinear control system*

$$\begin{array}{l}\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i(t)\, f_i(x) \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\}\end{array} \tag{3.24}$$

where U is a compact and convex subset of $\mathbb{R}^m$ and $f_0, \ldots, f_m$ are C^2 vector fields on $\mathbb{R}^d$. We assume, that following properties are satisfied.

(1) The nonlinear control system (3.24) has one singular point $x^* = 0 \in \mathbb{R}^d$, and (3.24) is Lie-determined such that $\mathbb{R}^d \setminus \{0\}$ and $\{0\}$ are maximal integral manifolds.

(2) There are periodic control functions u^h and $u^s \in \mathcal{U}$ such that the associated Lyapunov exponents of the linearized systems have the following properties

$$\begin{array}{rcll} & 0 & > \lambda_1^s \geq \ldots \geq \lambda_d^s & \text{and} \\ \lambda_1^h \geq \ldots \geq \lambda_k^h > & 0 & > \lambda_{k+1}^h \geq \ldots \geq \lambda_d^h & \text{for } 1 \leq k < d. \end{array}$$

Then define $\hat{\varepsilon}$ as in Notation 3.2.10, choose $\varepsilon \in (0, \hat{\varepsilon}]$ and denote by $\mathcal{X}_\varepsilon^{loc}, \mathcal{Y}_\varepsilon^{loc}$ the corresponding local unstable and stable fibre bundle of the differential equation (3.24) corresponding to u^h.

Moreover, suppose that one of the following two conditions are satisfied:

(3a) There is a neighborhood $V \subset \mathbb{R}^d$ of x^* such that for every $t \in \mathbb{R}$ and every $x \in \mathcal{X}_\varepsilon^{loc}(t) \cap V \setminus \{x^*\}$ the pair $(u^h(t + \cdot), x)$ is a strong inner pair.

(3b) There is a neighborhood $W \subset \mathbb{R}^d$ of x^* such that for every $t \in \mathbb{R}$ and every $x \in W \setminus \{x^*\}$ the pair $(u^s(t + \cdot), x)$ is a strong inner pair.

If (3a) is satisfied, then for every $\tau \in \mathbb{R}$ there exists a control set $D_\tau \subset \mathbb{R}^d$ with nonvoid interior and a $p_\tau \in \mathcal{X}_\varepsilon^{loc}(\tau) \cap V \setminus \{x^*\}$ with

$$\{\varphi(t, 0, p_\tau, u^h(\tau + \cdot)) : t < 0\} \subset \operatorname{int} D_\tau.$$

In particular we have $x^* \in \operatorname{cl} D_\tau$. If in addition all the pairs $\{(u^s(t+\cdot), \varphi(t, 0, p_\tau, u^s)), t \geq 0\}$ are strong inner pairs, then we also have

$$\{\varphi(t, 0, p_\tau, u^s) : t \geq 0\} \subset \operatorname{int} D_\tau.$$

If (3b) is satisfied, then for every $\tau \in \mathbb{R}$ there exists a control set $D_\tau \subset \mathbb{R}^d$ with nonvoid interior and a $p_\tau \in \mathcal{X}_\varepsilon^{loc}(\tau, u^h) \cap V \setminus \{x^*\}$ with

$$\{\varphi(t, 0, p_\tau, u^s) : t \geq 0\} \subset \operatorname{int} D_\tau.$$

In particular we have $x^* \in \operatorname{cl} D_\tau$. If in addition all the pairs $\{(u^s(t+\tau+\cdot), \varphi(t, 0, p_\tau, u^h(\tau+\cdot)), t < 0\}$ are strong inner pairs, then we also have

$$\{\varphi(t, 0, p_\tau, u^h(\tau + \cdot)) : t < 0\} \subset \operatorname{int} D_\tau.$$

Proof. Choose $\varepsilon \in (0, \hat{\varepsilon}]$ and $V^s \subset W(\varepsilon)$ as in Section 3.4 such that $V^s \subset V$. Next choose the open neighborhood $W^s \subset V^s$ of 0 with (3.20) and define $\hat{\rho} = \hat{\rho}(\varepsilon, W^s)$ as in Section 3.4. Then for $\rho \in (0, \hat{\rho}]$ we can construct the control function u as in Section 3.5 for a given $p \in \mathcal{T}^{loc}_{\varepsilon,\rho}(\tau)$. By applying Theorem 3.6.1 we find a point $p^* \in \mathcal{T}^{loc}_{\varepsilon,\rho}(\tau)$ such that

$$\{\Phi_t(u^*, p^*) : t \in \mathbb{R}\} \subset \omega(u, p)$$

where $u^* : \mathbb{R} \to U$ is defined by

$$u^*(t) := \begin{cases} u^s(t) & \text{for } t \geq 0, \\ u^h(t+\tau) & \text{for } t < 0. \end{cases}$$

Note that for every $-\infty < T_1 \leq T_2 < \infty$ the sets

$$\{\Phi_t(u^*, p^*) : t \in [T_1, T_2]\} \subset \mathcal{U} \times \mathbb{R}^d$$

are compact.

Now suppose that (3a) is fulfilled. Then choose $-\infty < T_0 < 0$ arbitrarily and a $\sigma > 0$ such that $T_0 + \sigma < 0$. Both $\Phi_{T_0}(u^*, p^*)$ and $\Phi_{T_0+\sigma}(u^*, p^*)$ are compact subsets of $\mathcal{U} \times \mathbb{R}^d$ and are strong inner pairs by assumption, because $\varphi(t, 0, p^*, u^h(\tau + \cdot)) \in V^s \subset W(\varepsilon) \cap V$ for all $t \leq 0$. Thus by applying Proposition 1.1.21 we find a control set $D \subset \mathbb{R}^d$ with nonvoid interior and

$$\varphi(T_0, 0, p^*, u^h(\tau + \cdot)) \subset \operatorname{int} D.$$

We show, that $\varphi(t, 0, p^*, u^h(\tau + \cdot)) \in \operatorname{int} D$ for all $t < 0$. Choose $-\infty < T_1 < T_2 < 0$ and $\sigma > 0$ with $T_0 \in [T_1, T_2]$. Then by applying Propositions 1.1.21 again, we find a control set $\tilde{D} \subset \mathbb{R}^d$ with $\{\varphi(t, 0, p^*, u^h(\tau + \cdot)) : t \in [T_1, T_2]\} \subset \operatorname{int} \tilde{D}$. On the other hand, since $\varphi(T_0, 0, p^*, u^h(\tau + \cdot)) \subset \operatorname{int} D$ we have $D \cap \tilde{D} \neq \emptyset$. By the maximality property of the control sets (cf. Definition 1.1.5 *(iii)*) we get $D = \tilde{D}$.

Next we suppose, that all the pairs $\{(u^s(t+\cdot), \varphi(t, 0, p_\tau, u^s)), t \geq 0\}$ are strong inner pairs. By choosing now $T_2 > 0$, we again get by applying Proposition 1.1.21 that there is a control set $\tilde{D} \subset \mathbb{R}^d$ with $\{\varphi(t, 0, p^*, u^*) : t \in [T_0, T_1]\} \subset \operatorname{int} \tilde{D}$. By maximality of control sets we finally obtain $D = \tilde{D}$.

The proof for (3b) works in the same way. ■

Remark 3.7.2 *Note, that the control set D_τ is not unique for a given $\tau \in \mathbb{R}$. The point $p^* \in \mathcal{T}^{loc}_{\varepsilon,\rho}(\tau)$ in the proof, which we get by the construction in Theorem 3.6.1, is not specified any further (cf. Remark 3.6.2). In the case where $\dim \mathcal{X}^{loc}_{\varepsilon,\rho} = 1$, the target $\mathcal{T}^{loc}_{\varepsilon,\rho}(\tau)$ consist of only two points, and thus the control sets D_τ can be described in more detail. This will be done in the Chapter 4.*

Figure 3.5 illustrates the result of the Theorem 3.7.1. If we assume here (for better illustration), that $\dim \mathcal{X}^{loc}_{\varepsilon} = 1$, then for a given $\tau \in \mathbb{R}$, the fibre $\mathcal{X}^{loc}_{\varepsilon}(\tau)$ is just a

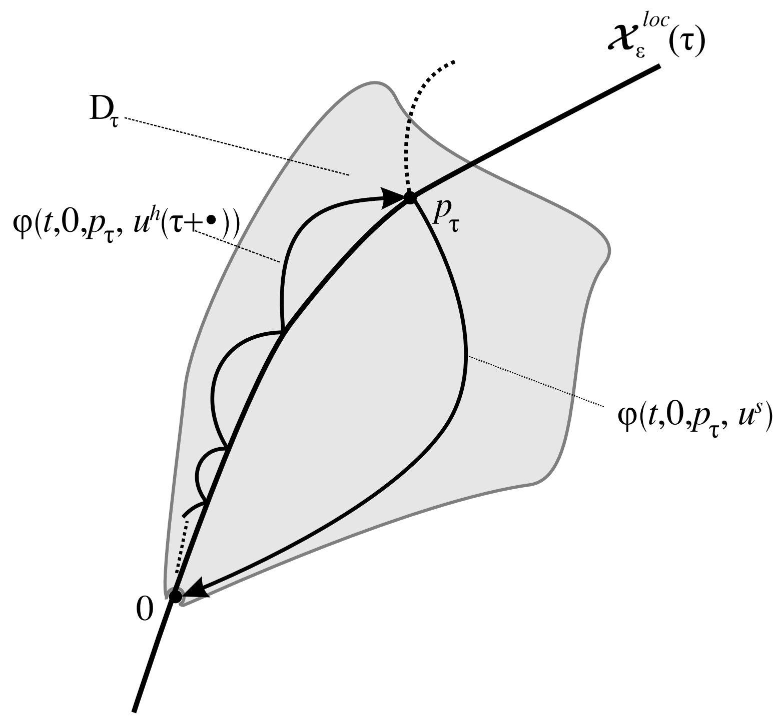

Figure 3.5: Illustration of the Existence Theorem

continuous curve in $\mathbb{R}^d$ through 0. We assume here, that the conditions (3a) and (3b) of the Theorem 3.7.1 are fulfilled. Now the result of the Theorem is, that we can find a point p_τ on the local unstable fibre bundle, such that if we apply the control function $u^*(\cdot, \tau)$ (defined as in Theorem 3.6.1), then the corresponding trajectory lies in the interior of a control set D_τ.

Note that we stated the Theorem without using the target $\mathcal{T}_{\varepsilon,\rho}^{loc}$. But it is clear, that for every $\rho > 0$ small enough, we get a corresponding control set $D_{\tau,\rho}$ with $\mathcal{T}_{\varepsilon,\rho}^{loc}(\tau) \cap D_{\tau,\rho} \neq 0$. The trajectory $\varphi(t, 0, p_\tau, u^*(\cdot, \tau))$ seems to jump out of the local unstable fibre bundle $\mathcal{X}_\varepsilon^{loc}(\tau)$ for $t < 0$. This is due to the fact, that we have drawn here only the fibre $\mathcal{X}_\varepsilon^{loc}(\tau)$ for a given $\tau \in \mathbb{R}$. For all $t < 0$ we have

$$\begin{aligned}\varphi(t, 0, p_\tau, u^*(\cdot, \tau)) &= \varphi(t, 0, p_\tau, u^h(\tau + \cdot)) \\ &= \varphi(t + \tau, \tau, p_\tau, u^h) \in \mathcal{X}_\varepsilon^{loc}(t + \tau).\end{aligned}$$

Because $\mathcal{X}_\varepsilon^{loc}(t)$ is 2Θ-periodic, we get for all $k \in \mathbb{N}$

$$\varphi(-2k\Theta + \tau, \tau, p_\tau, u^h) \in \mathcal{X}_\varepsilon^{loc}(-2k\Theta + \tau) = \mathcal{X}_\varepsilon^{loc}(\tau).$$

This explains the jumps. $\varphi(t, 0, p_\tau, u^h(\tau + \cdot))$ is an element of $\mathcal{X}_\varepsilon^{loc}(t + \tau)$ but we have drawn here only $\mathcal{X}_\varepsilon^{loc}(\tau)$. Thus for all $k \in \mathbb{N}$ the trajectory $\varphi(-2k\Theta + \tau, \tau, p_\tau, u^h)$ hits $\mathcal{X}_\varepsilon^{loc}(\tau)$ and for all other times it does not have to lie on $\mathcal{X}_\varepsilon^{loc}(\tau)$.

Chapter 4

The Onedimensional Case

In the previous section we assumed, that our nonlinear control system has a periodic control function u^h, such that the Lyapunov exponents $\lambda_1^h, \dots, \lambda_d^h$ of the corresponding linearized system fulfill

$$\lambda_1^h \geq \dots \geq \lambda_n^h > 0 > \lambda_{n+1}^h \geq \dots \geq \lambda_d^h \text{ for a } n \in \{1, \dots, d-1\}.$$

Now in this chapter we suppose, that $n = 1$, i.e.

$$\lambda_1^h > 0 > \lambda_2^h \geq \dots \geq \lambda_d^h, \tag{4.1}$$

which we call the *onedimensional case.* If one considers the differential equation

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i^h(t) f_u(x)$$

then the fibres of the local unstable fibre bundle $\mathcal{X}_\varepsilon^{loc}$ are onedimensional, i.e. they are continuous curves in $\mathbb{R}^d$. The fibres of the local stable fibre bundles $\mathcal{Y}_\varepsilon^{loc}$ are (topological) hyperplanes. The perturbed Duffing-van der Pol oscillator, which will be analyzed in Section 5.1, is an example for a system satisfying property (4.1).

For every $\tau \in \mathbb{R}$, the fibre $\mathcal{X}_\varepsilon^{loc}(\tau)$ of the unstable fibre bundle can be represented by two curves, which emerge from the singular point. By Theorem 3.7.1 of the previous section, we get the existence of one control set, where a subset of one of the two curves lies in its interior. Now the question arises, if there is also a control set with nonvoid interior which intersects the other curve. The answer to this question is yes and will be given in Theorem 4.1.12. As in the previous section, it relies on the construction of a control function $u \in \mathcal{U}$, but the construction here is different from the construction in Section 3.5. Here we will use the fact, that the fibres $\mathcal{Y}_\varepsilon^{loc}(\tau)$ of the (local) stable fibre bundle divide the state space into two disjoint subsets.

If the Lyapunov spectrum of the linearized system has the property

$$\begin{aligned} &\Sigma_{Ly} = \Sigma_{Ly}(\mathcal{V}_1) \oplus \Sigma_{Ly}(\mathcal{V}_2) \text{ with} \\ &0 \in \operatorname{int} \Sigma_{Ly}(\mathcal{V}_1) \text{ and } \Sigma_{Ly}(\mathcal{V}_2) \subset \mathbb{R}^- \end{aligned} \tag{4.2}$$

where $\mathcal{U} \times \mathbb{R}^d = \mathcal{V}_1 \oplus \mathcal{V}_2$ for two invariant subbundles $\mathcal{V}_1$ and $\mathcal{V}_2$ with

$$\dim \mathcal{V}_1 = 1 \tag{4.3}$$

then condition (4.1) is satisfied not only for one control function. In fact, we can find infinitely many of such control functions. For each of them we obtain the existence of two corresponding control sets. We will analyze, how these control sets, which belong to different u^h are related to each other (cf. Theorem 4.2.5). Here we will again use the idea of constructing an appropriate control function $u \in \mathcal{U}$.

Finally we investigate the relation between the control sets of the nonlinear control system near the singular point and the control sets of the linearized system. We consider a special case, where in addition to (4.2) and (4.3) we also assume, that

$$\Sigma_{Ly}(\mathcal{V}_1) \cap \Sigma_{Ly}(\mathcal{V}_2) = \emptyset.$$

Then the linearized control system exhibits two control sets, which are cones in $\mathbb{R}^d$. It turns out, that the control sets of the nonlinear system are close in a Lipschitz sense to the control sets of the linearized system (Theorem 4.3.5).

4.1 The Basic Construction Idea

In this section we will introduce some notations and the basic construction idea, which will be used in this chapter. We show, that if the dimension of the unstable fibre bundles is one, then we get two control sets, which have nonvoid intersection with the unstable fibre bundles.

4.1.1 Preliminaries

We consider the nonlinear control system on $\mathbb{R}^d$

$$\begin{array}{l} \dot{x} = f_0(x) + \sum_{i=1}^m u_i(t) f_i(x) \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a } t \in \mathbb{R}, \text{locally integrable}\}, \end{array} \tag{4.4}$$

where U is a compact and convex subset of $\mathbb{R}^m$ and $f_0, \dots, f_m$ are C^2 vector fields on $\mathbb{R}^d$. Furthermore, suppose that for all $(u, x) \in \mathcal{U} \times \mathbb{R}^d$ the equation (4.4) has a unique solution $\varphi(t, \tau, x, u), t, \tau \in \mathbb{R}$, with $\varphi(\tau, \tau, x, u) = x$.

We suppose, that the system (4.4) has the singular point $x^* = 0 \in \mathbb{R}^d$. Note that the assumption $x^* = 0$ is no restriction. If our control affine system has a singular point $x^* \neq 0$ we can transform the control system by an affine transformation into a control affine system with 0 as singular point. The assumption $x^* = 0$ is made here only for notational convenience.

Associated with the nonlinear system (4.4) is the bilinear control system on $\mathbb{R}^d$:

$$\begin{array}{l} \dot{x} = A_0 x + \sum_{i=1}^m u_i(t) A_i x \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\} \end{array} \tag{4.5}$$

where $A_i := \left.\frac{\partial f_i}{\partial x}\right|_{x=0}$. We denote the fundamental solution of (4.5) for $u \in \mathcal{U}$ by $\eta(t,\tau,u)$, with $\eta(\tau,\tau,u)x = x$, where $\tau, t \in \mathbb{R}, x \in \mathbb{R}^d$.

For the rest of this section, we suppose, that the following conditions are fulfilled.

Condition 4.1.1 *We assume, that there are two periodic control functions $u^h, u^s \in \mathcal{U}$ with the following properties:*

(a) *The control function u^h has period $\Theta \geq 0$ and the Lyapunov exponents $\lambda_1^h, \ldots, \lambda_d^h$ of the corresponding linear system*

$$\dot{x} = A_0 x + \sum_{i=1}^{m} u_i^h(t) A_i x$$

have the property

$$\lambda_1^h > 0 > \lambda_2^h \geq \ldots \geq \lambda_d^h. \tag{4.6}$$

(b) *For the control function u^s the corresponding linear system*

$$\dot{x} = A_0 x + \sum_{i=1}^{m} u_i^s(t) A_i x$$

has the Lyapunov exponents $\lambda_1^s, \ldots, \lambda_d^s$ with the property

$$0 > \lambda_1^s \geq \lambda_2^s \geq \ldots \geq \lambda_d^s. \tag{4.7}$$

We call u^h the hyperbolic control function *and u^s the* stable control function .

The superscript ^{h} indicates *hyperbolic* and the superscript ^{s} indicates *stable* as in the previous chapter.

If we apply to the nonlinear differential equation

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i^h(t) f_i(x) \tag{4.8}$$

the same reduction process as in Section 3.2, we get the reduced system

$$\begin{aligned} \dot{x} &= Ax + F_\varepsilon^+(t,x,y) \\ \dot{y} &= By + F_\varepsilon^-(t,x,y) \end{aligned} \tag{4.9}$$

on $X \times Y$. Thus by the choice of ε^* as in 3.2.1, the reduced system (4.9) has an unstable fibre bundle $\mathcal{X}_\varepsilon$ and a stable fibre bundle $\mathcal{Y}_\varepsilon$ for $\varepsilon \in (0, \varepsilon^*]$ and the hyperbolic system (4.8) has local unstable and stable fibre bundles $\mathcal{X}_\varepsilon^{loc}, \mathcal{Y}_\varepsilon^{loc}$.

We suppose that the following condition is fulfilled.

Condition 4.1.2 *The nonlinear control system (4.4) is locally accessible on* $\mathbb{R}^d \setminus \{0\}$*, and there is a neighborhood* $V \subset \mathbb{R}^d$ *such that for all* $\varepsilon \in (0, \hat{\varepsilon}], t \in \mathbb{R}$ *and all* $x \in \mathcal{Y}_\varepsilon^{loc}(t) \cap V$ *the pair* (u^h, x) *is a strong inner pair.*

The reason for the strong inner pair condition is, that we want to steer trajectories away from the stable fibre bundle $\mathcal{Y}_\varepsilon^{loc}$.

The basic idea in Section 3.5 was to construct a control function u by switching between u^h and u^s such that $\{\varphi(t, 0, p, u) : t \geq 0\}$ is bounded for a given $p \in \mathcal{X}_\varepsilon^{loc}(\tau)$. Thus we got a point $p^* \in \mathcal{T}_{\varepsilon,\rho}^{loc}(\tau)$ and a control function u^*, defined by

$$u^*(t) := \begin{cases} u^s(t) & \text{for } t \geq 0, \\ u^h(t+\tau) & \text{for } t < 0, \end{cases}$$

such that

$$\{\varphi(t, 0, p^*, u^*) : t \in \mathbb{R}\} \subset \operatorname{int} D.$$

Because $\dim X = 1$ the fibre $\mathcal{X}_\varepsilon^{loc}(\tau)$ is a continuous curve for every $\tau \in \mathbb{R}$ and the target consists of two points

$$\begin{aligned} \mathcal{T}_{\varepsilon,\rho}^{loc}(t) &= \{x \in \mathbb{R}^d : x \in \mathcal{F}(t)\mathcal{H}^{-1}(t, T_\rho)\} \\ &= p_{\varepsilon,\rho,>}(t) \cup p_{\varepsilon,\rho,<}(t) \end{aligned}$$

with two continuous and 2Θ-periodic curves $p_{\varepsilon,\rho>}, p_{\varepsilon,\rho,<} : \mathbb{R} \to \mathbb{R}^d$ and $p_{\varepsilon,\rho,>}(t), p_{\varepsilon,\rho,<}(t) \in \mathcal{X}_\varepsilon^{loc}(t)$. The construction of the control function u in Section 3.5 does *not* specify, if

$$p^* = p_{\varepsilon,\rho,>}(\tau) \text{ or } p^* = p_{\varepsilon,\rho,<}(\tau).$$

We will now explain how we modify the construction of the control function u, such that we can guarantee, that $p^* = p_{\varepsilon,\rho,>}(\tau)$. A similar modification on u guarantees, that $p^* = p_{\varepsilon,\rho,<}(\tau)$.

We have to introduce the following notation (cf. Figure 4.1).

Definition 4.1.3 *For* $\varepsilon \in (0, \varepsilon^*]$ *define for the reduced system (3.11)*

$$\begin{aligned} \mathcal{X}_{\varepsilon,>} &:= \{(t, x, y) \in X \times Y : x > w_\varepsilon^-(t, y)\}, \\ \mathcal{X}_{\varepsilon,\geq} &:= \mathcal{X}_{\varepsilon,>} \cup \mathcal{Y}_\varepsilon, \\ \mathcal{X}_{\varepsilon,<} &:= \{(t, x, y) \in X \times Y : x < w_\varepsilon^-(t, y)\}, \\ \mathcal{X}_{\varepsilon,\leq} &:= \mathcal{X}_{\varepsilon,<} \cup \mathcal{Y}_\varepsilon, \end{aligned}$$

and for the nonlinear system (4.4)

$$\begin{aligned} \mathcal{X}_{\varepsilon,>}^{loc} &:= \{(t, p) \in \mathbb{R} \times \mathbb{R}^d : (t, \mathcal{F}(t)p) \in \mathcal{X}_{\varepsilon,>}\}, \\ \mathcal{X}_{\varepsilon,\geq}^{loc} &:= \mathcal{X}_{\varepsilon,>} \cup \mathcal{Y}_\varepsilon, \\ \mathcal{X}_{\varepsilon,<}^{loc} &:= \{(t, p) \in \mathbb{R} \times \mathbb{R}^d : (t, \mathcal{F}(t)p) \in \mathcal{X}_{\varepsilon,<}\}, \\ \mathcal{X}_{\varepsilon,\leq}^{loc} &:= \mathcal{X}_{\varepsilon,<} \cup \mathcal{Y}_\varepsilon. \end{aligned}$$

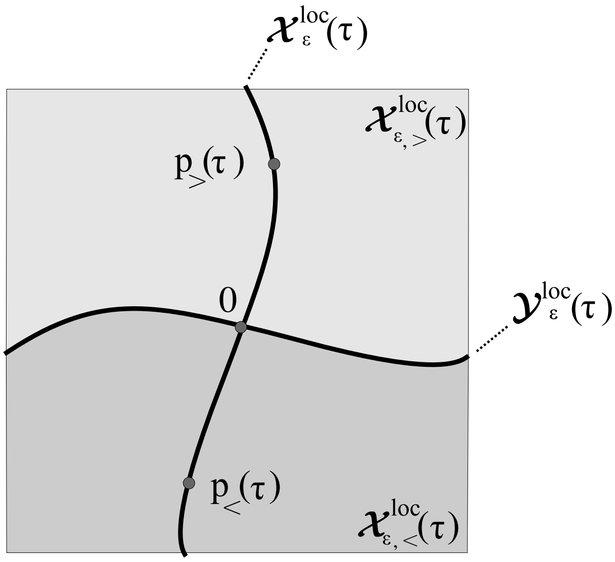

Figure 4.1: Sketch of the sets $\mathcal{X}^{loc}_{\varepsilon,>}$ and $\mathcal{X}^{loc}_{\varepsilon,<}$.

Note that for each $t \in \mathbb{R}$ we get a partition of $X \times Y$ into $\mathcal{X}_{\varepsilon,>}(t), \mathcal{X}_{\varepsilon,<}(t)$ and $\mathcal{Y}_{\varepsilon}(t)$ and a partition of $\mathbb{R}^d$ into $\mathcal{X}^{loc}_{\varepsilon,>}(t), \mathcal{X}^{loc}_{\varepsilon,<}(t)$ and $\mathcal{Y}^{loc}_{\varepsilon}(t)$. In addition $\mathcal{X}^{loc}_{\varepsilon,<}, \mathcal{X}^{loc}_{\varepsilon,>}$ and $\mathcal{Y}^{loc}_{\varepsilon}$ are connected subsets of $\mathbb{R}\times\mathbb{R}^d$, which are pairwise disjoint.

We choose and fix $\varepsilon \in (0, \hat{\varepsilon}]$ and $\rho \in (0, \hat{\rho}]$ and denote for abbreviation by $p_>(t)$ the curve with $p_>(t) \in \mathcal{X}^{loc}_{\varepsilon,>} \cap \mathcal{T}^{loc}_{\varepsilon,\rho}(t)$. We choose $p_>(\tau)$ as starting point for a $\tau \in \mathbb{R}$. The strategy is now to construct a control function $u_>$ such that

$$\varphi(t, \tau, p_>(\tau), u_>) \in \mathcal{X}^{loc}_{\varepsilon,\geq}(t) \text{ for all } t \geq \tau.$$

If we consider the construction of the control function as in Section 3.5, we can divide it into two parts. In the first part we steer with the stable control function u^s from a point near the target towards the origin, and in the second part we steer away from the origin with u^h. Here in this section, we also steer the trajectory with u^s from a point p near $p_>(\tau)$ towards the origin. But now there are two cases.

If the trajectory $\varphi(t, 0, p, u^s)$ does not hit the (local) stable fibre bundle $\mathcal{Y}^{loc}_{\varepsilon}(\tau)$, then $\varphi(t, 0, p, u^s) \in \mathcal{X}^{loc}_{\varepsilon,>}(\tau)$ for all $t \geq 0$. Thus by switching to u^h we get near $p_>(\tau)$ (cf. Figure 4.2 (a)).

If the trajectory $\varphi(t, 0, p, u^s)$ hits the stable fibre $\mathcal{Y}^{loc}_{\varepsilon}(\tau)$ at some time $T > 0$, then it may happen, that $\varphi(t, 0, p, u^s) \in \mathcal{X}^{loc}_{\varepsilon,<}(\tau)$ for all $t > T$ (cf. Figure 4.2 (b)). Thus if $\varphi(T, 0, p, u^s) \in \mathcal{Y}^{loc}_{\varepsilon}(\tau)$, then we apply the control function u^h to steer the trajectory towards the origin. Since we assumed, that (u^h, x) are strong inner pairs for all (u^h, x) with $x \in V \cap \mathcal{Y}^{loc}_{\varepsilon}(\tau)$, we can steer the trajectory into $\mathcal{X}^{loc}_{\varepsilon,>}(\tau)$. Finally we steer the trajectory with u^h towards $p_>(\tau)$.

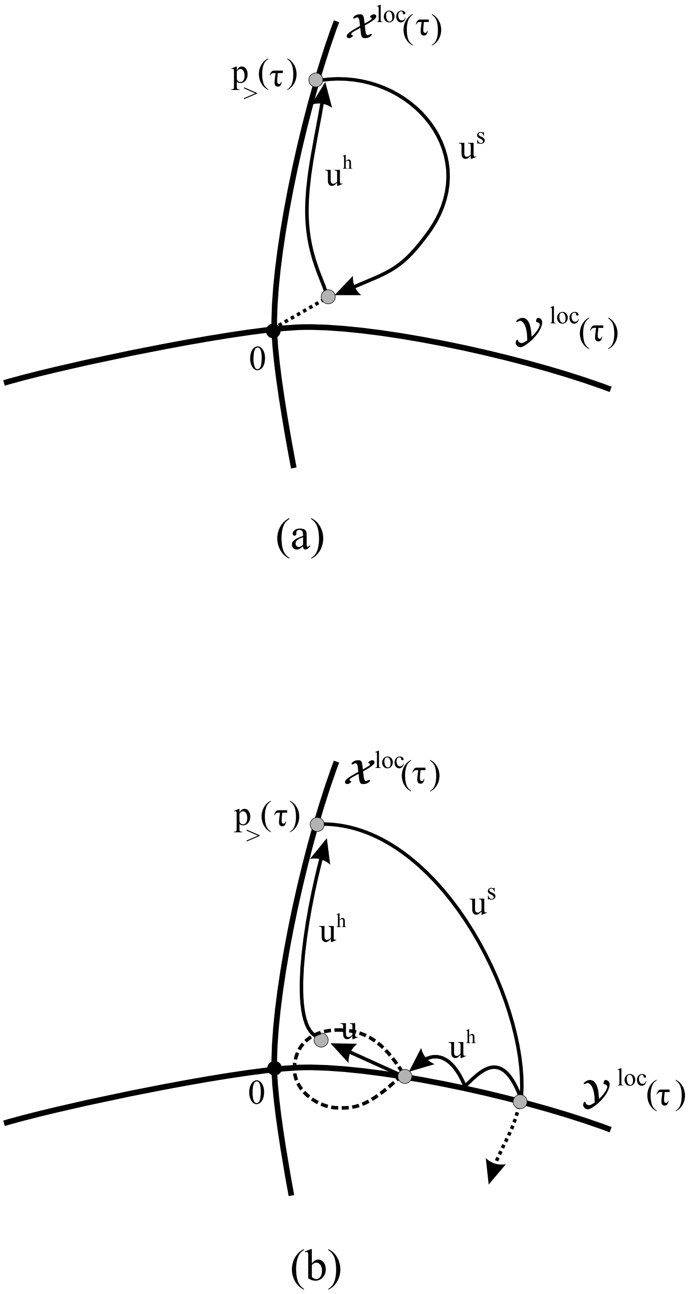

Figure 4.2: Construction idea of the control function $u_>$.

4.1.2 The Construction of the Control Function

We now construct the control function $u_>$ such that $p_{\varepsilon,\rho,>}(\tau) \in \pi_{\mathbb{R}^d}(\omega(u_>, p_{\varepsilon,\rho,>}(\tau)))$.

First we adjust the neighborhoods as in Section 3.4. For the definition of $\hat{\varepsilon}$ see Notation 3.2.10. We choose and fix an $\varepsilon \in (0, \hat{\varepsilon}]$, which means that there is a neighborhood $W(\varepsilon) \subset \mathbb{R}^d$ of 0 such that for all $p \in \mathcal{X}_\varepsilon^{loc}(\tau) \cap W(\varepsilon)$ and all $q \in \mathcal{Y}_\varepsilon^{loc}(\tau) \cap W(\varepsilon)$ we have

$$\begin{aligned} \varphi(t,\tau,p,u^h) &= \mathcal{F}^{-1}(t)\psi(t,\tau,\mathcal{F}(\tau)p,u^h) = \mathcal{F}^{-1}(t)\mu_\varepsilon(t,\tau,\mathcal{F}(\tau)p,u^h) \text{ for all } t \leq \tau, \\ \varphi(t,\tau,q,u^h) &= \mathcal{F}^{-1}(t)\psi(t,\tau,\mathcal{F}(\tau)q,u^h) = \mathcal{F}^{-1}(t)\mu_\varepsilon(t,\tau,\mathcal{F}(\tau)q,u^h) \text{ for all } t \geq \tau, \end{aligned} \tag{4.10}$$

according to Lemma 6.2.9.

Note that the system

$$\dot{x} = f_0(x) + \sum_{i=1}^m u_i^s(t) f_i(x) \tag{4.11}$$

is locally asymptotically stable (cf. Lemma 6.2.10). Thus we can choose a neighborhood $V^s \subset \mathbb{R}^d$ of 0 such that

$$V^s \subset W(\varepsilon)$$

and an open neighborhood $W^s \subset \mathbb{R}^d$ of 0, such that for all $p \in W^s$ we have

$$\varphi(t,0,p,u^s) \in V^s \text{ for all } t \geq 0 \text{ and } \lim_{t\to\infty} \varphi(t,0,p,u^s) = 0. \tag{4.12}$$

By Notation 3.2.10 there is a $\hat{\rho} := \hat{\rho}(\varepsilon, W^s) \in (0, \rho^*(\varepsilon))$ such that

$$\mathcal{D}_{\varepsilon,\rho}^{loc}(t) \subset W^s \text{ for all } \rho \in (0,\hat{\rho}) \text{ and all } t \in \mathbb{R}. \tag{4.13}$$

Thus for all $\rho \in (0, \hat{\rho})$ we have

$$\mathcal{T}_{\varepsilon,\rho}^{loc}(t) \subset W^s \subset V^s \subset W(\varepsilon) \text{ for all } t \in \mathbb{R}. \tag{4.14}$$

We choose and fix $\rho \in (0, \hat{\rho})$. For abbreviation we write $p_> := p_{\varepsilon,\rho,>}$ and $p_< := p_{\varepsilon,\rho,<}$.

Next we choose a sequence $(\delta_i)_{i\in\mathbb{N}} \subset \mathbb{R}^+$ with $\lim_{i\to\infty} \delta_i = 0$ and

$$B_{\delta_i}(\mathcal{D}_{\varepsilon,\rho}^{loc}) \subset W^s \text{ for all } i \in \mathbb{N}, t \in \mathbb{R}$$

and

$$B_{\delta_i}(p_>(\tau)) \cap B_{\delta_i}(p_<(\tau)) = \emptyset$$

which is possible because of relation (4.14).

According to Proposition 3.3.3 there exists a sequence $(\sigma_i)_{i\in\mathbb{N}} \subset \mathbb{R}^+$ with $\lim_{i\to\infty} \sigma_i = 0$, such that we have:

For every $\tau \in \mathbb{R}, q \in \mathcal{T}^{loc}_{\varepsilon,\rho}(\tau)$ there is a $K_i \in \mathbb{N}$ with

$$2K_i\Theta > 2^i$$

such that for every $k \geq K_i$, $k \in \mathbb{N}$ we have $\varphi(-2k\Theta + \tau, \tau, q, u^h) \in B_{\rho_i}(0)$. Furthermore, if $p \in B_{\sigma_i}(0)$ and $\mathcal{P}^{loc}_\varepsilon(\tau, p) = \varphi(-2k\Theta + \tau, \tau, q, u^h)$, then

$$\begin{array}{rcl} \varphi(t, \tau, p, u^h) & \in & B_{\delta_{i+1}}(\mathcal{X}^{loc}_\varepsilon(t)) \text{ for every } t \in [\tau, \tau + 2k\Theta], \text{ and} \\ \varphi(2k\Theta + \tau, \tau, p, u^h) & \in & B_{\delta_{i+1}}(\mathcal{T}^{loc}_{\varepsilon,\rho}(\tau)) = B_{\delta_{i+1}}(p_>(\tau)) \cup B_{\delta_{i+1}}(p_>(\tau)). \end{array} \tag{4.15}$$

We construct the control function $u_>$ by starting at

$$p := p_{-1} := p_>(\tau) \in \mathcal{T}^{loc}_{\varepsilon,\rho}(\tau), \tag{4.16}$$

which is by assumption an element of $\mathcal{X}^{loc}_{\varepsilon,>}(\tau)$.

Step 0, Part A:
There are $\Delta T_0 > 0$ and $\Delta T_1, \Delta T_2 \geq 0$ and a control function $u_0 \in \mathcal{U}$ such that

$$\begin{array}{ll} p_0 := \varphi(\Delta T_0, 0, p, u^s) & \in \mathcal{X}^{loc}_{\varepsilon,\geq}(\tau) \\ p_1 := \varphi(\Delta T_1, 0, p_0, u^h) & \in B_{\sigma_0}(0) \cap \mathcal{X}^{loc}_{\varepsilon,\geq}(\tau) \\ p_2 := \varphi(\Delta T_2, 0, p_1, u_0) & \in B_{\sigma_0}(0) \cap \mathcal{X}^{loc}_{\varepsilon,>}(\tau) \end{array}$$

with

$$\mathcal{P}^{loc}_\varepsilon(\tau, p_2) = \varphi(-2k_0\Theta + \tau, \tau, p_>(\tau), u^h).$$

Proof. There are two cases: either

$$\varphi(t, 0, p, u^s) \in \mathcal{X}^{loc}_{\varepsilon,>}(\tau) \text{ for all } t > 0$$

or there is a time $\Delta T_0 > 0$ with

$$\begin{array}{l} \varphi(\Delta T_0, 0, p, u^s) \in \mathcal{Y}^{loc}_\varepsilon(\tau) \text{ and} \\ \quad \varphi(t, 0, p, u^s) \in \mathcal{X}^{loc}_{\varepsilon,>}(\tau) \text{ for all } 0 < t < \Delta T_0. \end{array}$$

- If we are in the *first case*, by $p \in W^s$ there exists a time $\Delta \tilde{T}_0 > 1$ with

$$\varphi(t, 0, p, u^s) \in B_{\rho_0}(0) \ \cap \mathcal{X}^{loc}_{\varepsilon,>}(\tau) \text{ for all } t > \Delta \tilde{T}_0.$$

Since the projection mapping $\mathcal{P}^{loc}_\varepsilon$ is continuous with $\mathcal{P}^{loc}_\varepsilon(\tau, 0) = 0$ and because

$$\lim_{t\to\infty} \varphi(t, 0, p, u^s) = 0,$$

we get $\lim_{t\to\infty} \mathcal{P}^{loc}_\varepsilon(\tau, \varphi(t, 0, p, u^s)) = 0$. Note, that by relation (4.14) we can apply Proposition 3.2.11 and from (4.15), we get a $k_0 \in \mathbb{N}$ with $k_0 > K_0$ and

$$\Delta T_0 > \Delta \tilde{T}_0, q_0 \in \mathcal{T}^{loc}_{\varepsilon,\rho}(\tau)$$

with

$$\mathcal{P}_\varepsilon^{loc}(\tau, \varphi(\Delta T_0, 0, p, u^s)) = \varphi(-2k_0\Theta + \tau, \tau, q_0, u^h).$$

On the other hand since $\varphi(\Delta T_0, 0, p, u^s) \in \mathcal{X}_{\varepsilon,>}^{loc}(\tau)$ we have $q_0 = p_>(\tau) = p$. We define

$$\begin{aligned} p_0 &:= p_1 := p_2 := \varphi(\Delta T_0, 0, p, u^s) \text{ and} \\ \Delta T_1 &:= \Delta T_2 := 0, \end{aligned}$$

and choose a function $u_0 \in \mathcal{U}$ arbitrarily.

• In the *second case* we define

$$p_0 := \varphi(\Delta T_0, 0, p, u^s).$$

Because $p_0 \in \mathcal{Y}_\varepsilon^{loc}(\tau)$ there is a time $\Delta\widetilde{T}_1 > 0$ such that $\varphi(t + \tau, \tau, p_0, u^h) \in B_{\sigma_0}(0)$ for all $t \geq \Delta\widetilde{T}_1$. Now choose $\Delta T_1 > \Delta\widetilde{T}_1$ such that

$$\Delta T_1 = 2l_0\Theta \text{ for a } l_0 \in \mathbb{N}.$$

Thus

$$p_1 := \varphi(\Delta T_1 + \tau, \tau, p_0, u^h) = \varphi(\Delta T_1, 0, p_0, u^h(\tau + \cdot))$$

and by periodicity of $\mathcal{Y}_\varepsilon^{loc}(\cdot)$ we have $p_1 \in B_{\sigma_0}(0) \cap \mathcal{Y}_\varepsilon^{loc}(\Delta T_1 + \tau) = B_{\sigma_0}(0) \cap \mathcal{Y}_\varepsilon^{loc}(\tau)$. By assumption (u^h, p_1) is a strong inner pair, thus there exists a time $\Delta\widetilde{T}_2 < 1$ and a control function $u_0 \in \mathcal{U}$ such that

$$\begin{aligned} &\varphi(\Delta\widetilde{T}_2, 0, p_1, u_0) \in B_{\sigma_0}(0) \cap \mathcal{X}_{\varepsilon,>}^{loc}(\tau) \text{ and} \\ &\quad \varphi(t, 0, p_1, u_0) \in B_{\sigma_0}(0) \text{ for all } t \in [0, \Delta\tilde{T}_2]. \end{aligned}$$

Then there is an interval $[a_0, b_0] \subset [0, \Delta\tilde{T}_2]$ with

$$\begin{array}{rcll} \varphi(a_0, 0, p_1, u_0) & \in & \mathcal{Y}_\varepsilon^{loc}(\tau) & \text{and} \\ \varphi(t, 0, p_1, u_0) & \in & \mathcal{X}_{\varepsilon,>}^{loc}(\tau) & \text{for all } t \in (a_0, b_0]. \end{array}$$

From $\varphi(a_0, 0, p_1, u_0) \in \mathcal{Y}_\varepsilon^{loc}(\tau)$ we get $\mathcal{P}_\varepsilon^{loc}(\tau, \varphi(a_0, 0, p_1, u_0)) = 0$, and since

$$\varphi(t, 0, p_1, u_0) \notin \mathcal{Y}^{loc}(\tau) \text{ for all } t \in (a_0, b_0]$$

we get $\mathcal{P}_\varepsilon^{loc}(\tau, \varphi(t, 0, p_1, u_0)) \in \mathcal{X}_{\varepsilon,>}^{loc}(\tau) \setminus \{0\}$.

By Proposition 3.2.11 there is a $k_0 \in \mathbb{N}$ with $k_0 > K_0$ and $q_0 \in \mathcal{T}_{\varepsilon,\rho}^{loc}(\tau)$, $\Delta T_1 \in (a_0, b_0]$ with

$$\mathcal{P}_\varepsilon^{loc}(\tau, \varphi(\Delta T_1, 0, p_0, u_0)) = \varphi(-2k_0\Theta^h + \tau, \tau, q_0, u^h).$$

But from $\varphi(\Delta T_1, 0, p_0, u^s) \in \mathcal{X}_{\varepsilon,>}^{loc}(\tau)$ we have $q_0 = p_>(\tau)$. Define

$$p_2 := \varphi(\Delta T_2, 0, p_1, u_0).$$

■

Step 0, Part B:
There is a $k_0 > K_0$ such that with

$$\begin{array}{lll} \Delta T_3 & := & 2k_0\Theta \\ p_3 & := & \varphi(\Delta T_3 + \tau, \tau, p_2, u^h) = \varphi(\Delta T_3, 0, p_2, u^h(\tau + \cdot)) \end{array}$$

we get

$$p_3 \in B_{\delta_1}(p_>(\tau)) \subset \mathcal{X}^{loc}_{\varepsilon,>}(\tau).$$

Proof. In Step 0 Part A, we stopped with the point $p_2 \in B_{\sigma_0}(0) \cap \mathcal{X}^{loc}_{\varepsilon,>}(\tau)$ and

$$\mathcal{P}^{loc}_\varepsilon(\tau, p_2) = \varphi(-2k_0\Theta + \tau, \tau, p_>(\tau), u^h).$$

Define

$$\Delta T_3 := 2k_0\Theta$$

and

$$p_3 := \varphi(\Delta T_3 + \tau, \tau, p_2, u^h) = \varphi(\Delta T_3, 0, p_2, u^h(\tau + \cdot)).$$

Because of (4.15) we have $p_3 \in B_{\delta_1}(p_>(\tau)) \subset \mathcal{X}^{loc}_{\varepsilon,>}(\tau)$. ■

This is the end of Step 0. We define the times ΔT_i, the points $p_i \in \mathcal{X}^{loc}_{\varepsilon,>}(\tau)$ and the controls $u_i \in \mathcal{U}$ for $i = 1, 2, \dots$ recursively.

By construction we have $p_{4i-1} \in B_{\delta_i}(p_>(\tau))$ such that it is an element of $\mathcal{X}^{loc}_{\varepsilon,>}(\tau)$.
Step i, Part A:
There are $\Delta T_{4i} > 0$ and $\Delta T_{4i+1} \geq 0, \Delta T_{4i+2} \in (0, \frac{1}{2^i})$ and a control function $u_i \in \mathcal{U}$ such that

$$\begin{array}{lll} p_{4i} & := \varphi(\Delta T_{4i}, 0, p_{4i-1}, u^s) & \in \mathcal{X}^{loc}_{\varepsilon,\geq}(\tau) \\ p_{4i+1} & := \varphi(\Delta T_{4i+1}, 0, p_{4i}, u^h) & \in B_{\rho_i}(0) \cap \mathcal{X}^{loc}_{\varepsilon,\geq}(\tau) \\ p_{4i+2} & := \varphi(\Delta T_{4i+2}, 0, p_{4i+1}, u_i) & \in B_{\rho_i}(0) \cap \mathcal{X}^{loc}_{\varepsilon,>}(\tau) \end{array}$$

with

$$\mathcal{P}^{loc}_\varepsilon(\tau, p_{4i+2}) = \varphi(-2k_i\Theta + \tau, \tau, p_>(\tau), u^h) \text{ for a } k_i > K_i.$$

Proof. Either we have

$$\varphi(t, 0, p_{4i-1}, u^s) \in \mathcal{X}^{loc}_{\varepsilon,>}(\tau) \text{ for all } t > 0$$

or there is a time $\Delta T_{4i} > 0$ with

$$\begin{array}{l} \varphi(\Delta T_{4i}, 0, p_{4i-1}, u^s) \in \mathcal{Y}^{loc}_\varepsilon(\tau) \text{ and} \\ \quad \varphi(t, 0, p_{4i-1}, u^s) \in \mathcal{X}^{loc}_{\varepsilon,>}(\tau) \text{ for all } 0 < t < \Delta T_{4i}. \end{array}$$

- In the *first case*, because $p_{4i-1} \in W^s$ there exists a time $\Delta\tilde{T}_{4i} > 2^i$ with

$$\varphi(t, 0, p_{4i-1}, u^s) \in B_{\sigma_i}(0) \cap \mathcal{X}^{loc}_{\varepsilon,>}(\tau) \text{ for all } t > \Delta\tilde{T}_{4i}.$$

Since the projection mapping $\mathcal{P}^{loc}_{\varepsilon}$ is continuous and because $\lim_{t\to\infty} \varphi(t, 0, p_{4i-1}, u^s) = 0$, we get $\lim_{t\to\infty} \mathcal{P}^{loc}_{\varepsilon}(\tau, \varphi(t, 0, p_{4i-1}, u^s)) = 0$. By assumption there is a $k_i \in \mathbb{N}$ with $k_i > K_i$ and $\Delta T_{4i} > \Delta\tilde{T}_{4i}, q_i \in \mathcal{T}^{loc}_{\varepsilon,\rho}(\tau)$ with

$$\mathcal{P}^{loc}_{\varepsilon}(\tau, \varphi(\Delta T_{4i}, 0, p_{4i-1}, u^s)) = \varphi(-2k_i\Theta + \tau, \tau, q_i, u^h).$$

On the other hand since $\varphi(\Delta T_{4i}, 0, p_{4i-1}, u^s) \in \mathcal{X}^{loc}_{\varepsilon,>}(\tau)$ we have $q_i = p_>(\tau)$. We define

$$\begin{aligned} p_{4i} := p_{4i+1} := p_{4i+2} &:= \varphi(\Delta T_{4i}, 0, p_{4i-1}, u^s) \text{ and} \\ \Delta T_{4i+1} := \Delta T_{4i+2} &:= 0 \end{aligned}$$

and choose a function $u_i \in \mathcal{U}$ arbitrary.

- In the *second case* we define

$$p_{4i} := \varphi(\Delta T_{4i}, 0, p_{4i-1}, u^s).$$

Since $p_{4i} \in \mathcal{Y}^{loc}_{\varepsilon}(\tau)$ there is a time $\Delta\widetilde{T}_{4i+1} > 0$ such that $\varphi(t + \tau, \tau, p_{4i}, u^h) \in B_{\sigma_i}(0)$ for all $t \geq \Delta\widetilde{T}_{4i+1}$. Now choose $\Delta T_{4i+1} > \Delta\widetilde{T}_{4i+1}$ such that

$$\Delta T_{4i+1} = 2l_i\Theta \text{ for a } l_i \in \mathbb{N}.$$

Define

$$p_{4i+1} := \varphi(\Delta T_{4i+1} + \tau, \tau, p_{4i}, u^h) = \varphi(\Delta T_{4i+1}, 0, p, u^h(\tau + \cdot))$$

and note that $p_{4i+1} \in B_{\sigma_i}(0) \cap \mathcal{Y}^{loc}_{\varepsilon}(\tau)$ by periodicity of $\mathcal{Y}^{loc}_{\varepsilon}(\cdot)$. By assumption (u^h, p_{4i+1}) is a strong inner pair, thus there exists a time $\Delta\tilde{T}_{4i+2} < \frac{1}{2^i}$ and a control function $u_i \in \mathcal{U}$ such that

$$\begin{aligned} \varphi(\Delta\widetilde{T}_{4i+2}, 0, p_{4i+1}, u_i) &\in B_{\sigma_i}(0) \cap \mathcal{X}^{loc}_{\varepsilon,>}(\tau) \text{ and} \\ \varphi(t, 0, p_{4i+1}, u_i) &\in B_{\sigma_i}(0) \text{ for all } t \in [0, \Delta\tilde{T}_{4i+2}]. \end{aligned}$$

Then there is an interval $[a_i, b_i] \subset [0, \Delta\tilde{T}_{4i+2}]$ with

$$\begin{aligned} \varphi(a_i, 0, p_{4i+1}, u_i) &\in \mathcal{Y}^{loc}_{\varepsilon}(\tau) \text{ and} \\ \varphi(t, 0, p_{4i+1}, u_i) &\in \mathcal{X}^{loc}_{\varepsilon,>}(\tau) \text{ for all } t \in (a_i, b_i]. \end{aligned}$$

Since $\varphi(a_i, 0, p_{4i+1}, u_i) \in \mathcal{Y}^{loc}_{\varepsilon}(\tau)$ we get $\mathcal{P}^{loc}_{\varepsilon}(\tau, \varphi(a_i, 0, p_{4i+1}, u_i)) = 0$, and because $\varphi(t, 0, p_{4i+1}, u_i) \notin \mathcal{Y}^{loc}_{\varepsilon}(\tau)$ we get $\mathcal{P}^{loc}_{\varepsilon}(\tau, \varphi(t, \tau, p_{4i+1}, u_i)) \in \mathcal{X}^{loc}_{\varepsilon}(\tau) \setminus \{0\}$ for $t \in (a_i, b_i]$. By Proposition 3.2.11 there is a $k_i \in \mathbb{N}$ such that $k_i > K_i$ and $q_i \in \mathcal{T}^{loc}_{\varepsilon,\rho}(\tau)$, $\Delta T_{4i+2} \in (a_i, b_i]$ with

$$\mathcal{P}^{loc}_{\varepsilon}(\tau, \varphi(\Delta T_{4i+2}, 0, p_{4i+1}, u_i)) = \varphi(-2k_i\Theta + \tau, \tau, q_i, u^h).$$

On the other hand since $\varphi(\Delta T_{4i+2}, 0, p_{4i+1}, u^s) \in \mathcal{X}^{loc}_{\varepsilon,>}(\tau)$ we have $q_i = p_>(\tau)$. Define

$$p_{4i+2} := \varphi(\Delta T_{4i+2}, 0, p_{4i+1}, u_i).$$

■

Step i, Part B:
There is a $k_i > K_i$ such that with

$$\begin{aligned} \Delta T_{4i+3} &:= 2k_i\Theta \\ p_{4i+3} &:= \varphi(\Delta T_{4i+3} + \tau, \tau, p_{4i+2}, u^h) = \varphi(\Delta T_{4i+3}, 0, p_{4i+2}, u^h(\tau + \cdot)) \end{aligned}$$

we get

$$p_{4i+3} \in B_{\delta_{i+1}}(p_>(\tau)) \subset \mathcal{X}^{loc}_{\varepsilon,>}(\tau)$$

.

Proof. In Step i, Part A we stopped with a point $p_{4i+2} \in B_{\sigma_i}(0) \cap \mathcal{X}^{loc}_{\varepsilon,>}(\tau)$ and

$$\mathcal{P}^{loc}_\varepsilon(\tau, p_{4i+2}) = \varphi(-2k_i\Theta + \tau, \tau, p_>(\tau), u^h).$$

We define

$$\begin{aligned} &\Delta T_{4i+3} := 2k_i\Theta \text{ and} \\ &p_{4i+3} := \varphi(\Delta T_{4i+3} + \tau, \tau, p_{4i+2}, u^h) = \varphi(\Delta T_{4i+3}, 0, p_{4i+2}, u^h(\tau + \cdot)). \end{aligned}$$

From (4.15) we obtain

$$p_{4i+3} \in B_{\delta_{i+1}}(p_>(\tau)) \subset \mathcal{X}^{loc}_{\varepsilon,>}(\tau). \tag{4.17}$$

■

This is the end of the construction algorithm. Now define

$$T_i := \sum_{k=0}^{i} \Delta T_k$$

and the function $u_> : \mathbb{R} \to U$ by

$$u_>(t) := \begin{cases} 0, & \text{for } t < 0, \\ u^s(t - T_{4i-1}) & \text{for } t \in [T_{4i-1}, T_{4i}), \\ u^h(t + \tau - T_{4i}) & \text{for } t \in [T_{4i}, T_{4i+1}), \\ u_i(t - T_{4i+1}) & \text{for } t \in [T_{4i+1}, T_{4i+2}), \\ u^h(t + \tau - T_{4i+2}) & \text{for } t \in [T_{4i+2}, T_{4i+3}), \end{cases}$$

for $i = 0, 1, 2, \ldots$ where we define $T_{-1} := 0$.

Remark 4.1.4 *By the construction given above, the trajectory* $\varphi(t,\tau,p_>(\tau),u_>)$ *lies in* $\mathcal{X}^{loc}_{\varepsilon,\geq}(\tau)$ *for all* $t \geq \tau$. *If we want to construct the corresponding control function* $u_<$ *such that* $\varphi(t,\tau,p_<(\tau),u_<) \in \mathcal{X}^{loc}_{\varepsilon,\leq}(\tau)$ *for all* $t \geq \tau$, *then we have to exchange all occurrences of* $\mathcal{X}^{loc}_{\varepsilon,>}(\tau)$ *by* $\mathcal{X}^{loc}_{\varepsilon,<}(\tau)$ *in the procedure above. Furthermore we have to start at the point* $p_<(\tau)$.

Remark 4.1.5 *Here we have to make the same remarks as at the end of Section 3.5, Remark 3.5.1 and Remark 3.5.2. The control function* $u_>$ *depends on the constants* ε, ρ, τ. *In Chapter 3, it was possible that* $\dim X > 1$, *i.e. the target* $\mathcal{T}^{loc}_{\varepsilon,\rho}(\tau)$ *could be a higher dimensional object, such that we were really able to* choose *the starting point* p_{-1}. *But here, in the case* $\dim X = 1$, *we have no real choice, since there are only two points in* $\mathcal{T}_{\varepsilon,\rho}(\tau)$. *Deciding that the starting point* p *has to be an element of* $\mathcal{X}^{loc}_{\varepsilon,>}$ *this point has to be* $p_>(\tau)$. *Thus to specify* $u_>$ *more detailed we may write* $u_{>,\varepsilon,\rho,\tau} := u_>$.

4.1.3 The Limit Set

We now want to show, that $p_>(\tau) \in \pi_{\mathbb{R}^d}(\omega(u_>,p_>(\tau)))$. In Section 3.6 we also have found the $\mathcal{U}$-component of $\omega(u_>,p_>(\tau))$, which belongs to $p_>(\tau)$ in $\mathcal{U} \times \mathbb{R}^d$. Because the construction of $u_>$ is here different from the construction in Section 3.5, the $\mathcal{U}$-component can also look different. We have to distinguish two cases. Either we have $\varphi(t,0,p_>(\tau),u^s) \in \mathcal{X}^{loc}_{\varepsilon,>}(\tau)$ for all $t > 0$ or not.

We will first consider the case $\varphi(t,0,p_>(\tau),u^s) \in \mathcal{X}^{loc}_{\varepsilon,>}(\tau)$ for all $t > 0$.

Lemma 4.1.6 *Let* $\tau \in \mathbb{R}$ *and assume, that we have*

$$\varphi(t,0,p_>(\tau),u^s) \in \mathcal{X}^{loc}_{\varepsilon,>}(\tau) \text{ for all } t > 0.$$

Let $u_> \in \mathcal{U}$ *be constructed as in Section 4.1.2. Then for the times* ΔT_{4i} *and* ΔT_{4i+3} *of the construction we have*

$$\Delta T_{4i} \to \infty \text{ and } \Delta T_{4i+3} \to \infty \text{ for } i \to \infty.$$

Proof. Suppose, that $\Delta T_{4i} < N$ for a $N > 0$. By continuous dependence on initial values there is an $i_0 \in \mathbb{N}$ such that $\varphi(\Delta T_{4i},0,p,u^s) \notin B_{\sigma_i}(0)$ for all $p \in B_{\delta_i}(p_>(\tau))$ and all $i > i_0$. Thus by the construction of the control function $u_>$ it follows that $p_{4i} = \varphi(\Delta T_{4i},0,p_{4i-1},u^s) \in \mathcal{Y}^{loc}_{\varepsilon}(\tau)$ for all $i > i_0$. This is a contradiction to the assumption $\varphi(t,0,p_>(\tau),u^s) \notin \mathcal{Y}^{loc}_{\varepsilon}(\tau)$ for all $t > 0$. Therefore the sequence $(\Delta T_{4i})_{i\in\mathbb{N}}$ is unbounded.

With $\Delta T_{4i+3} = 2k_i\Theta > 2^i$ the assertion follows. ■

After this preparation, we can characterize the ω-limit set in the first case (cf. Figure 4.3).

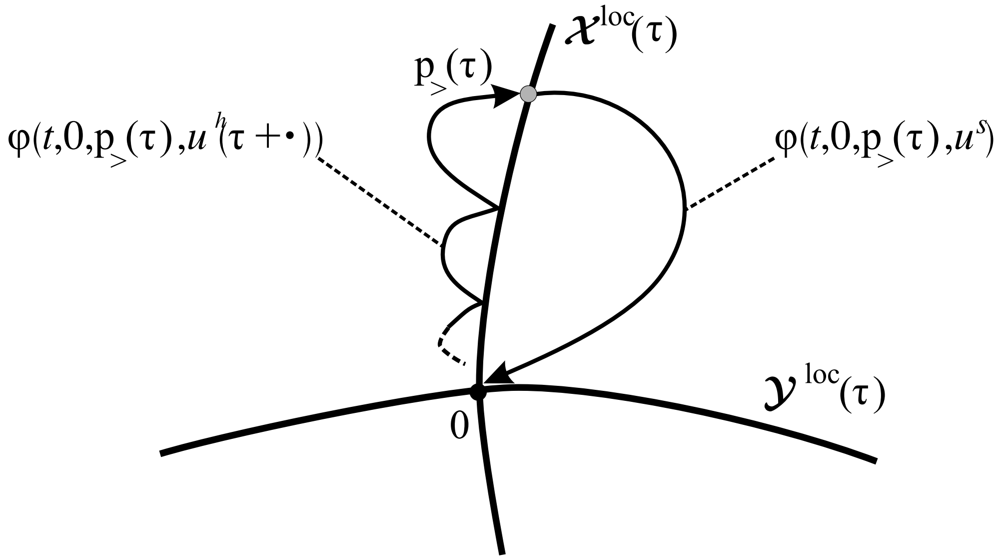

Figure 4.3: Sketch of $\varphi(t, 0, p_>(\tau), u^*)$ if $\varphi(t, 0, p_>(\tau), u^s) \in \mathcal{X}^{loc}_{\varepsilon,>}(\tau)$ for all $t > 0$.

Proposition 4.1.7 *Let $\tau \in \mathbb{R}$ with $\varphi(t, 0, p_>(\tau), u^s) \in \mathcal{X}_{\varepsilon,>}^{loc}(\tau)$ for all $t > 0$ and let $u_> \in \mathcal{U}$ be constructed as in Section 4.1.2. Then we have*

$$\{\Phi_t(u^*, p_>(\tau)) : t \in \mathbb{R}\} \subset \omega(u_>, p_>(\tau))$$

where the control function $u^* \in \mathcal{U}$ is defined by

$$u^*(t) := \begin{cases} u^s(t) & \text{for } t \geq 0, \\ u^h(t+\tau) & \text{for } t < 0. \end{cases}$$

Proof. We show, that

$$\lim_{k\to\infty} \theta_{T_{4k-1}} u_> = u^*$$

with the same technique as in the proof of Theorem 3.6.1. It suffices to show that for every $g \in L^1(\mathbb{R}, \mathbb{R}^m)$ and $\sigma > 0$ there is a $N \in \mathbb{N}$ such that

$$\left| \int_{\mathbb{R}} \langle u_>(t + T_{4k-1}) - u^*(t), g(t) \rangle \, dt \right| < \sigma \text{ for all } k > N.$$

Let $g \in L^1(\mathbb{R}, \mathbb{R}^m)$ and $\sigma > 0$. Then there exists a time $T > 0$ with

$$\int_{\mathbb{R}\setminus[-T,T]} |g(t)| \, dt < \frac{\sigma}{2\mathrm{diam} U}.$$

Furthermore, by Lemma 4.1.6, there exists a $N \in \mathbb{N}$ with

$$\Delta T_{4k-1} > T \text{ and}$$
$$\Delta T_{4k} > T \text{ for all } k > N.$$

This guarantees, that

$$\begin{aligned} u_>(t+T_{4k-1}) &= u^s(t) && \text{for all } t \in [0,T] \text{ and} \\ u_>(t+T_{4k-1}) &= u^h(t+\tau) && \text{for all } t \in [-T,0]. \end{aligned}$$

It follows

$$\begin{aligned} &\left| \int_{\mathbb{R}} \langle u_>(t+T_{4k-1}) - u^*(t), g(t) \rangle \, dt \right| \\ &\leq \left| \int_{\mathbb{R}^-} \langle u_>(t+T_{4k-1}) - u^*(t), g(t) \rangle \, dt \right| + \left| \int_{\mathbb{R}^+} \langle u_>(t+T_{4k-1}) - u^*(t), g(t) \rangle \, dt \right| \\ &\leq \left| \int_{\mathbb{R}^- \setminus [-T,0]} \langle u_>(t+T_{4k-1}) - u^*(t), g(t) \rangle \, dt \right| + \left| \int_{[-T,0]} \langle u_>(t+T_{4k-1}) - u^*(t), g(t) \rangle \, dt \right| \\ &+ \left| \int_{\mathbb{R}^+ \setminus [0,T]} \langle u_>(t+T_{4k-1}) - u^*(t), g(t) \rangle \, dt \right| + \left| \int_{[0,T]} \langle u_>(t+T_{4k-1}) - u^*(t), g(t) \rangle \, dt \right| \\ &< \frac{\sigma}{2} + \left| \int_{[-T,0]} \left\langle u^h(t+\tau) - u^*(t), g(t) \right\rangle dt \right| + \frac{\sigma}{2} + \left| \int_{[0,T]} \langle u^s(t) - u^*(t), g(t) \rangle \, dt \right| \\ &< \frac{\sigma}{2} + 0 + \frac{\sigma}{2} + 0 \\ &< \sigma \end{aligned}$$

Thus we have shown $\lim_{k\to\infty} \theta_{T_{4k-1}} u_> = u^*$.

By relation (4.17) we have $p_{4k-1} = \varphi(T_{4k-1}, 0, p_>(\tau), u_>) \in B_{\delta_k}(p_>(\tau))$ and we get

$$p_>(\tau) = \lim_{k\to\infty} \varphi(T_{4k-1}, 0, p_>(\tau), u_>).$$

Thus it follows, that $(u^*, p_>(\tau)) \in \omega(u_>, p_>(\tau))$. By invariance of $\omega(u_>, p_>(\tau))$ the assumption follows. ■

Remark 4.1.8 *We obtain a similar result for $u_<$: Choose $\tau \in \mathbb{R}$ with $\varphi(t, 0, p_<(\tau), u^s) \in \mathcal{X}^{loc}_{\varepsilon,<}(\tau)$ for all $t > 0$. If we construct $u_< \in \mathcal{U}$ as in Section 4.1.2, we get*

$$\{\Phi_\tau(u^*, p_<(\tau)) : \tau \in \mathbb{R}\} \subset \omega(u_<, p_<(\tau)).$$

Now consider the case where there is a time $\xi > 0$ such that

$$\begin{aligned} &\varphi(\xi, 0, p_>(\tau), u^s) \in \mathcal{Y}^{loc}_\varepsilon(\tau) \text{ and} \\ &\varphi(t, 0, p_>(\tau), u^s) \in \mathcal{X}^{loc}_{\varepsilon,>}(\tau) \text{ for all } t \in [0, \xi]. \end{aligned}$$

It turns out, that in this case, the ω-limit set can have a slightly different structure than in the case before.

Lemma 4.1.9 *Let $\tau \in \mathbb{R}$ and let $u_> \in \mathcal{U}$ be constructed as in Section 4.1.2. Assume, that there is a time $\xi > 0$ such that*

$$\begin{aligned} &\varphi(\xi, 0, p_>(\tau), u^s) \in \mathcal{Y}_\varepsilon^{loc}(\tau) \text{ and} \\ &\varphi(t, 0, p_>(\tau), u^s) \in \mathcal{X}_{\varepsilon>}^{loc}(\tau) \text{ for all } t \in [0, \xi]. \end{aligned} \tag{4.18}$$

Then

$$\liminf_{i \to \infty} \Delta T_{4i} = \xi.$$

Furthermore, we have

$$\Delta T_{4i+3} \to \infty \text{ for } i \to \infty.$$

Proof. Suppose, that there is a $\gamma \in (0, \xi)$ and a subsequence of $(\Delta T_{4i_k})_{k \in \mathbb{N}}$ with $\Delta T_{4i_k} \leq \xi - \gamma$ for all $k \in \mathbb{N}$. For simplicity, denote this subsequence by $(\Delta T_{4i})_{i \in \mathbb{N}}$. Since this sequence is bounded, there exists a convergent subsequence, which we will again denote by $(\Delta T_{4i})_{i \in \mathbb{N}}$ with $\lim_{i \to \infty} \Delta T_{4i} = \hat{T} \in [0, \xi - \gamma]$. On the other hand from $p_{4i-1} \in B_{\delta_i}(p_>(\tau))$ we have $\lim_{i \to \infty} \varphi(\Delta T_{4i}, 0, p_{4i-1}, u^s) = \varphi(\hat{T}, 0, p_>(\tau), u^s) \in \mathcal{Y}_\varepsilon^{loc}(\tau)$, which is a contradiction. This shows the first assertion.

Because $\Delta T_{4i+3} = 2k_i \Theta > 2^i$ the second assertion follows. ■

Proposition 4.1.10 *Let $\tau \in \mathbb{R}$ and let $u_> \in \mathcal{U}$ be constructed as in Section 4.1.2. Assume, that there is a time $\xi > 0$ such that*

$$\begin{aligned} &\varphi(\xi, 0, p_>(\tau), u^s) \in \mathcal{Y}_\varepsilon^{loc}(\tau) \text{ and} \\ &\varphi(t, 0, p_>(\tau), u^s) \in \mathcal{X}_{\varepsilon,>}^{loc}(\tau) \text{ for all } t \in [0, \xi]. \end{aligned}$$

Then at least one of the following two cases occur:

(a) We have

$$\{\Phi_t(u^*, p_>(\tau)) : t \in \mathbb{R}\} \subset \omega(u_>, p_>(\tau))$$

where the function $u^* \in \mathcal{U}$ defined by

$$u^*(t) := \begin{cases} u^s(t) & \text{for } t \geq 0, \\ u^h(t + \tau) & \text{for } t < 0. \end{cases}.$$

(b) There is a $\zeta \geq \xi$ such that with the control function $v^* \in \mathcal{U}$ defined by

$$v^*(t) := \begin{cases} u^h(t + \tau) & \text{for } t < 0, \\ u^s(t) & \text{for } t \in [0, \zeta], \\ u^h(t + \tau - \zeta) & \text{for } t > \zeta. \end{cases}$$

we have

$$\{\Phi_t(v^*, p_>(\tau)) : t \in \mathbb{R}\} \subset \omega(u_>, p_>(\tau)).$$

(cf. Figure 4.4)

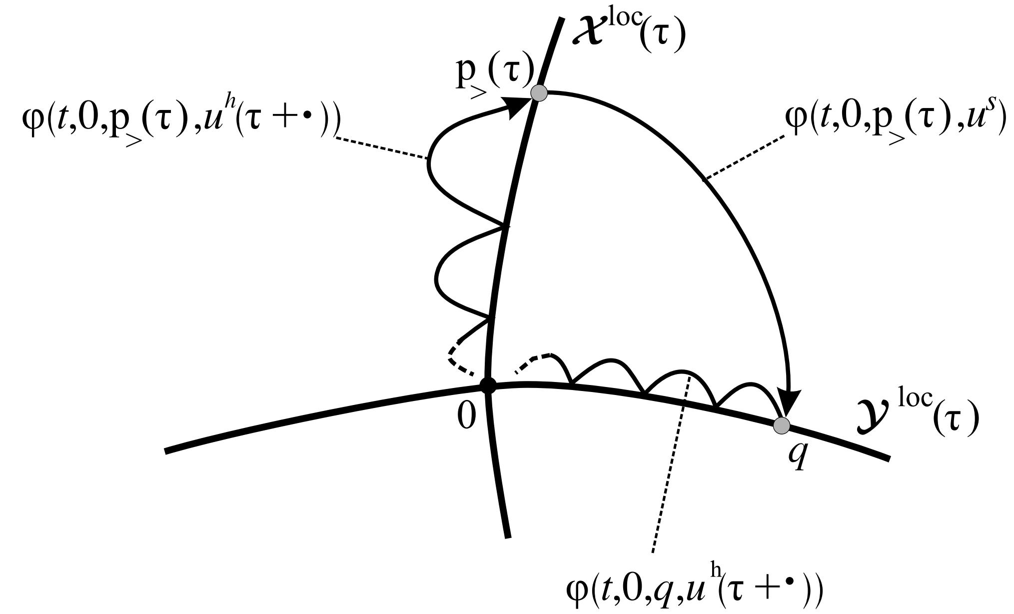

Figure 4.4: Sketch of $\varphi(t, 0, p_>(\tau), v^*)$ if there is a $\xi > 0$ with $\varphi(\xi, 0, p_>(\tau), u^s) \in \mathcal{Y}_\varepsilon^{loc}(\tau)$.

Proof. We have to consider two different cases. There may be a subsequence $(\Delta T_{4i_l})_{l\in\mathbb{N}}$ with $\Delta T_{4i_l} \to \infty$ for $l \to \infty$ *or* there may be a subsequence $(\Delta T_{4i_m})_{m\in\mathbb{N}}$ which is bounded. For simplification we will denote the subsequences by $(\Delta T_i)_{i\in\mathbb{N}}$.

- *Case 1:* $\Delta T_{4i} \to \infty$ *for* $i \to \infty$.

As in the proof of Proposition 4.1.7, we get $\{\Phi_t(u^*, p_>(\tau)) : t \in \mathbb{R}\} \subset \omega(u_>, p_>(\tau))$.

- *Case 2: The sequence* $(\Delta T_{4i})_{i\in\mathbb{N}}$ *is bounded.*

By construction of the control function, this means there is an $i_0 \in \mathbb{N}$ with

$$\varphi(\Delta T_{4i}, 0, p_{4i-1}, u^s) \in \mathcal{Y}_\varepsilon^{loc}(\tau)$$

for all $i > i_0$. Otherwise, there would be a subsequence $(\Delta T_{4i_l})_{l\in\mathbb{N}}$ with

$$\varphi(\Delta T_{4i_l}, 0, p_{4i_l-1}, u^s) \in \mathcal{X}_{\varepsilon,>}^{loc}(\tau).$$

But by construction of the control function $u_>$ then $\varphi(\Delta T_{4i_l}, 0, p_{4i_l-1}, u^s) \in B_{\sigma_{i_l}}(0)$ which is a contradiction to the assumption, that ΔT_{4i_l} is bounded and $p_{4i_l-1} \in B_{\delta_{i_l}}(p_>(\tau))$.

Since the sequence is $(\Delta T_{4i})_{i\in\mathbb{N}}$ is bounded, it has a convergent subsequence, which we will denote again by $(\Delta T_{4i})_{i\in\mathbb{N}}$. By Lemma 4.1.9 we get $\liminf_{i\to\infty} \Delta T_{4i} = \xi$, and we obtain

$$\lim_{i\to\infty} \Delta T_{4i} =: \zeta \geq \xi.$$

Since $p_{4i-1} \in B_{\delta_i}(p_>(\tau))$ and $\varphi(\Delta T_{4i}, 0, p_{4i-1}, u^s) \in \mathcal{Y}_\varepsilon^{loc}(\tau)$ we get by continuous dependency on initial conditions, that there is a $q^* \in \mathcal{Y}_\varepsilon^{loc}(\tau)$ with

$$\lim_{i\to\infty} \varphi(\Delta T_{4i}, 0, p_{4i-1}, u^s) = \lim_{i\to\infty} p_{4i} = q^* \in \mathcal{Y}_\varepsilon^{loc}(\tau).$$

Furthermore we get

$$\Delta T_{4i+1} \to \infty \text{ for } i \to \infty.$$

Otherwise suppose that $\Delta T_{4i+1} < N$ for a $N > 0$. Because $\lim_{i\to\infty} p_{4i} = q^*$ this is a contradiction to the construction of the control function, where we have chosen ΔT_{4i+1} in such a way, that $p_{4i+1} = \varphi(\Delta T_{4i+1} + \tau, \tau, p_{4i}, u^h) \in B_{\sigma_i}(0)$.

Define the control function $v^* \in \mathcal{U}$ by

$$v^*(t) := \begin{cases} u^h(t) & \text{for } t < 0, \\ u^s(t) & \text{for } t \in [0, \zeta], \\ u^h(t + \tau - \zeta) & \text{for } t > \zeta. \end{cases}$$

We show, that

$$\lim_{i\to\infty} \theta_{T_{4i}} u_> = v^*.$$

It suffices to show that for every $g \in L^1(\mathbb{R}, \mathbb{R}^m)$ and $\sigma > 0$ there is a $N \in \mathbb{N}$ such that

$$\left| \int_{\mathbb{R}} \langle u_>(t + T_{4i-1}) - v^*(t), g(t) \rangle \, dt \right| < \sigma \text{ for all } i > N.$$

Let $g \in L^1(\mathbb{R}, \mathbb{R}^m)$ and $\sigma > 0$. Then there exists a time $T > \zeta$ with

$$\int_{\mathbb{R}\setminus[-T,T]} |g(t)| \, dt < \frac{\sigma}{4 \mathrm{diam} U}.$$

Because $\lim_{i\to\infty} \Delta T_{4i} = \zeta$ there exists a $N_1 \in \mathbb{N}$ with

$$\begin{aligned} \Delta T_{4i} &< T \text{ and} \\ \Delta T_{4i-1} &> T \text{ for all } i > N_1. \end{aligned} \tag{4.19}$$

This guarantees, that

$$u_>(t + T_{4i-1}) = u^s(t) \text{ for all } t \in [0, \Delta T_{4i}], \tag{4.20}$$

$$u_>(t + T_{4i-1}) = u^h(t + \tau - \Delta T_{4i}) \text{ for all } t \in [\Delta T_{4i}, T] \text{ and} \tag{4.21}$$

$$u_>(t + T_{4i-1}) = u^h(t + \tau) \text{ for all } t \in [-T, 0]. \tag{4.22}$$

First we show, that there is a $N_2 \in \mathbb{N}, N_2 > N_1$ such that

$$\left| \int_{[\Delta T_{4i}, T]} \langle u_>(t + T_{4i-1}) - v^*(t), g(t) \rangle \, dt \right| < \frac{\sigma}{4} \text{ for all } i > N_2. \tag{4.23}$$

We have

$$\begin{aligned} &\left| \int_{[\Delta T_{4i}, T]} \langle u_>(t + T_{4i-1}) - v^*(t), g(t) \rangle \, dt \right| \\ &\leq \left| \int_{[\zeta, \Delta T_{4i}]} \langle u_>(t + T_{4i-1}) - v^*(t), g(t) \rangle \, dt \right| + \left| \int_{[\Delta T_{4i}, T]} \langle u_>(t + T_{4i-1}) - v^*(t), g(t) \rangle \, dt \right| \end{aligned}$$

where we define $[\zeta, \Delta T_{4i}] := \emptyset$ if $\zeta > \Delta T_{4i}$.

For the first integral in the sum we have because $\lim_{i\to\infty} \Delta T_{4i} = \zeta$

$$\lim_{i\to\infty} \left| \int_{[\zeta,\Delta T_{4i}]} \langle u_>(t + T_{4i}) - v^*(t), g(t)\rangle \, dt \right| = 0$$

For the second integral in the sum we can therefore suppose, that $\Delta T_{4i} \geq \zeta$. We define for an interval $[a, b] \subset \mathbb{R}$ the characteristic function

$$\chi_{[a,b]}(t) := \begin{cases} 1 & \text{for } t \in [a, b], \\ 0 & \text{for } t \notin [a, b]. \end{cases}$$

Thus we get

$$\begin{aligned} & \left| \int_{[\Delta T_{4i},T]} \langle u_>(t + T_{4i-1}) - v^*(t), g(t)\rangle \, dt \right| \\ = & \left| \int_{\mathbb{R}} \chi_{[\Delta T_{4i},T]}(t) \, \langle [u_>(t + T_{4i-1}) - v^*(t)], g(t)\rangle \, dt \right| \\ = & \left| \int_{\mathbb{R}} \chi_{[\Delta T_{4i},T]}(t) \langle [u^h(t + \tau - \Delta T_{4i-1}) - u^h(t + \tau - \zeta)], g(t)\rangle dt \right| \end{aligned}$$

From

$$u^h(t + \tau - \Delta T_{4i}) = \theta_{\zeta - \Delta T_{4i}} u^h(t + \tau - \zeta)$$

and since $(\mathcal{U}, \theta)$ is a continuous dynamical system it follows

$$\lim_{i\to\infty} u^h(\cdot + \tau - \Delta T_{4i}) = u^h(\cdot + \tau - \zeta).$$

Now with $\lim_{i\to\infty} \chi_{[\Delta T_{4i},T]}(\cdot) = \chi_{[\zeta,T]}(\cdot)$ it follows

$$\lim_{i\to\infty} \left| \int_{\mathbb{R}} \chi_{[\Delta T_{4i},T]}(t) \left\langle [u^h(t + \tau - \Delta T_{4i-1}) - u^h(t + \tau - \zeta)], g(t) \right\rangle dt \right| = 0$$

Thus we can choose $N_2 > N_1$ such that (4.23) is satisfied.

Furthermore, because $\lim_{i\to\infty} \Delta T_{4i} = \zeta$ and (4.20) there exists a $N > N_2$ with

$$\int_{[0,\Delta T_{4i}]} |\langle u_>(t + T_{4i-1}) - v^*(t), g(t)\rangle| \, dt < \frac{\sigma}{4}.$$

Then we get for $i > N$:

$$
\begin{aligned}
&\left|\int_{\mathbb{R}} \langle u_>(t+T_{4i-1}) - v^*_>(t), g(t)\rangle \, dt\right| \\
&\le \left|\int_{\mathbb{R}^-} \langle u_>(t+T_{4i-1}) - v^*(t), g(t)\rangle \, dt\right| + \left|\int_{\mathbb{R}^+} \langle u_>(t+T_{4i-1}) - v^*(t), g(t)\rangle \, dt\right| \\
&\le \left|\int_{\mathbb{R}^-\setminus[-T,0]} \langle u_>(t+T_{4i-1}) - v^*(t), g(t)\rangle \, dt\right| + \left|\int_{[-T,0]} \langle u_>(t+T_{4i-1}) - v^*(t), g(t)\rangle \, dt\right| \\
&+ \left|\int_{\mathbb{R}^+\setminus[0,T]} \langle u_>(t+T_{4i-1}) - v^*(t), g(t)\rangle \, dt\right| + \left|\int_{[0,T]} \langle u_>(t+T_{4i-1}) - v^*(t), g(t)\rangle \, dt\right| \\
&< \frac{\sigma}{4} + \left|\int_{[-T,0]} \left\langle u^h(t+\tau) - v^*(t), g(t)\right\rangle dt\right| + \frac{\sigma}{4} + \left|\int_{[0,T]} \langle u_>(t+T_{4i-1}) - v^*(t), g(t)\rangle \, dt\right| \\
&< \frac{\sigma}{4} + 0 + \frac{\sigma}{4} + \left|\int_{[0,\Delta T_{4i}]} \langle u_>(t+T_{4i-1}) - v^*(t), g(t)\rangle \, dt\right| \\
&+ \left|\int_{[\Delta T_{4i},T]} \langle u_>(t+T_{4i-1}) - v^*(t), g(t)\rangle \, dt\right| \\
&< \sigma
\end{aligned}
$$

Hence we have shown $\lim_{i\to\infty} \theta_{T_{4i-1}} u_> = v^*$.

From relation (4.17) we have $p_{4i-1} = \varphi(T_{4i-1}, 0, p_>(\tau), u_>) \in B_{\delta_i}(p_>(\tau))$ and we obtain

$$p_>(\tau) = \lim_{i\to\infty} \varphi(T_{4i-1}, 0, p_>(\tau), u_>).$$

Thus it follows, that $(v^*, p_>(\tau)) \in \omega(u_>, p_>(\tau))$. Finally, from invariance of $\omega(u_>, p_>(\tau))$ (cf. Corollary 1.1.16) we obtain, that $(\theta_t v^*, \varphi(t, 0, p_>(\tau), v^*)) \in \omega(u_>, p_>(\tau))$ for all $t \in \mathbb{R}$. ■

Remark 4.1.11 *Again one has to mention, that this proposition holds if we replace* $p_>(\tau)$ *by* $p_<(\tau)$*,* $u_>$ *by* $u_<$ *and* $\mathcal{X}^{loc}_{\varepsilon,>}(\tau)$ *by* $\mathcal{X}^{loc}_{\varepsilon,<}(\tau)$*.*

4.1.4 The Existence Theorem

Since we have now contructed the control function $u_>$ in Section 4.1.2 and characterized the ω-limit set in Section 4.1.3, we are now in the position to state the following theorem.

Theorem 4.1.12 *Consider the nonlinear control system*

$$
\begin{aligned}
&\dot{x} = f_0(x) + \textstyle\sum_{i=1}^m u_i(t) f_i(x) \\
&u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \ \textit{for a.a. } t \in \mathbb{R}, \textit{locally integrable}\}
\end{aligned}
\tag{4.24}
$$

where U is a compact and convex subset of $\mathbb{R}^m$ and $f_0, \dots, f_m$ are C^2 vector fields on $\mathbb{R}$. Suppose that for all $(u,x) \in \mathcal{U} \times \mathbb{R}^d$ the equation (4.24) has a unique solution $\varphi(t,\tau,x,u), t, \tau \in \mathbb{R}$, with $\varphi(\tau,\tau,x,u) = x$. Assume, that following properties are satisfied.

(1) The nonlinear control system (4.24) has a singular point $x^* = 0 \in \mathbb{R}^d$, and (4.24) is Lie-determined such that $\mathbb{R}^d \setminus \{0\}$ and $\{0\}$ are maximal integral manifolds.

(2) There are two piecewise constant and periodic control functions u^h and $u^s \in \mathcal{U}$ such that the associated Lyapunov exponents of the linearized systems have the following properties

$$\begin{aligned} 0 &> \lambda_1^s \geq \dots \geq \lambda_d^s \text{ and} \\ \lambda_1^h > 0 &> \lambda_2^h \geq \dots \geq \lambda_d^h. \end{aligned}$$

For $\varepsilon \in (0,\hat{\varepsilon}]$ denote by $\mathcal{X}_\varepsilon^{loc}, \mathcal{Y}_\varepsilon^{loc}$ the corresponding local unstable and stable fibre bundle of the differential equation corresponding to u^h. Note that $\hat{\varepsilon}$ is defined as in Notation 3.2.10.

Moreover, suppose that there is a neighborhood $V \subset \mathbb{R}^d$ of x^* such that

(3) for every $t \in \mathbb{R}$ and every $x \in \mathcal{X}_{\hat{\varepsilon}}^{loc}(t) \cap V \setminus \{0\}$ the pair $(u^h(t+\cdot), x)$ is a strong inner pair.

(4) for every $t \in \mathbb{R}$ and every $x \in \mathcal{Y}_{\hat{\varepsilon}}^{loc}(t) \cap V \setminus \{0\}$ the pair $(u^h(t+\cdot), x)$ is a strong inner pair.

(5) for every $t \in \mathbb{R}$ and every $x \in V \setminus \{0\}$ the pair $(u^s(t+\cdot), x)$ is a strong inner pair.

Then there exist two control sets $D_>, D_< \subset \mathbb{R}^d$ with nonvoid interior such that for every $\varepsilon \in (0,\hat{\varepsilon}]$ there is a neighborhood $W \subset \mathbb{R}^d$ such that for every $\tau \in \mathbb{R}$ we have

$$\{\varphi(t,0,p_>,u^h(\tau+\cdot)) : t < 0\} \subset \operatorname{int} D_> \tag{4.25}$$

for every $p_> \in \mathcal{X}_{\varepsilon,>}^{loc}(\tau) \cap W \setminus \{0\}$ and

$$\{\varphi(t,0,p_<,u^h(\tau+\cdot)) : t < 0\} \subset \operatorname{int} D_< \tag{4.26}$$

for every $p_< \in \mathcal{X}_{\varepsilon,<}^{loc}(\tau) \cap W \setminus \{0\}$. In particular, we have

$$x^* \in \operatorname{cl} D_> \cap \operatorname{cl} D_<. \tag{4.27}$$

Furthermore for $p_> \in \mathcal{X}_{\varepsilon,>}^{loc}(\tau) \cap W \setminus \{0\}$ we get two cases: Either

$$\{\varphi(t,0,p_>,u^s) : t \geq 0\} \subset \operatorname{int} D_> \tag{4.28}$$

or there is a $\zeta > 0$ *with*

$$\begin{array}{l}\varphi(\zeta, 0, p_>, u^s) \in \mathcal{Y}_\varepsilon^{loc}(\tau),\\ \{\varphi(t, 0, p_>, u^s) : t \in [0, \zeta]\} \subset \operatorname{int} D_> \text{ and}\\ \{\varphi(t, 0, \varphi(\zeta, p, u^s), u^h(\tau - \zeta + \cdot)), t \geq \zeta\} \subset \operatorname{int} D_>.\end{array} \tag{4.29}$$

The statements (4.28) or (4.29) hold true for every $p_< \in \mathcal{X}_{\varepsilon,<}^{loc}(\tau) \cap W \setminus \{x^*\}$ *if we replace* $D_<$ *by* $D_>$.

Proof. We first have to adjust the neighborhoods. Choose $\varepsilon \in (0, \hat{\varepsilon}]$, choose the neighborhoods V^s and W^s of 0 as in Section 4.1.2 and define $\hat{\rho} := \hat{\rho}(\varepsilon, W^s)$ such that the relation (4.14) is fulfilled.

We show the existence of $D_>$ with the properties (4.25) and (4.27), the construction of $D_<$ works in the same way.

Fix $\rho \in (0, \hat{\rho}]$ and $\tau \in \mathbb{R}$ and define $p_{-1} := p_>(\tau) = \mathcal{T}_{\varepsilon,\rho}^{loc}(\tau) \cap \mathcal{X}_{\varepsilon,>}^{loc}(\tau)$. By the construction 4.1.2, we get a control function $u_> \in \mathcal{U}$. According to Proposition 4.1.7 and Proposition 4.1.10 there are two cases: Either we have

$$\{\Phi_t(u^*, p_>(\tau)) : t \in \mathbb{R}\} \subset \omega(u_>, p_>(\tau))$$

with $u^* \in \mathcal{U}$ defined by

$$u^*(t) := \begin{cases} u^s(t) & \text{for } t \geq 0, \\ u^h(t + \tau) & \text{for } t < 0, \end{cases}$$

or there is a $\zeta > 0$ such that with the control function $v^* \in \mathcal{U}$ defined by

$$v^*(t) := \begin{cases} u^h(t + \tau) & \text{for } t < 0, \\ u^s(t) & \text{for } t \in [0, \zeta], \\ u^h(t + \tau - \zeta) & \text{for } t > \zeta. \end{cases}$$

we get

$$\{\Phi_t(v^*, p_>(\tau)) : t \in \mathbb{R}\} \subset \omega(u_>, p_>(\tau))$$

and $\varphi(\zeta, p_>, u^s) \in \mathcal{Y}_\varepsilon^{loc}(\tau)$.

In the first case, because of the premises (3),(4) and (5) for every $-\infty < T_1 < 0 < T_2 < \infty$ and $\gamma > 0$ the compact sets

$$\begin{array}{l}\{\Phi_t(u^*, p_>(\tau)) : t \in [T_1, T_2]\} \text{ and}\\ \{\Phi_t(u^*, p_>(\tau)) : t \in [T_1 + \gamma, T_2 + \gamma]\}\end{array}$$

consist of strong inner pairs. Thus by applying Proposition 1.1.21 we get a control set $D(p_>(\tau)) \subset \mathbb{R}^d$ with

$$\{\varphi(t, 0, p_>(\tau), u^*) : t \in [T_1, T_2]\} \subset \operatorname{int} D(p_>(\tau)).$$

By maximality of control sets it follows

$$\{\varphi(t,0,p_>(\tau),u^*) : t \in \mathbb{R}\} \subset \operatorname{int} D(p_>(\tau)).$$

In the second case because of the premises (3),(4) and (5) for every $-\infty < T_1 < 0 < T_2 < \infty$ and $\gamma > 0$ the compact sets

$$\begin{array}{ll} \{\Phi_t(v^*, p_>(\tau)) : t \in [T_1, T_2]\} & \text{and} \\ \{\Phi_t(v^*, p_>(\tau)) : t \in [T_1+\gamma, T_2+\gamma]\} & \end{array}$$

also consist of strong inner pairs and we get in the same way that

$$\{\varphi(t,0,p_>(\tau),v^*) : t \in \mathbb{R}\} \subset \operatorname{int} D(p_>(\tau)).$$

Next we show, that for every

$$p_1, p_2 \in \mathcal{X}^{loc}_{\varepsilon,>}(\tau) \setminus \{0\}$$

with

$$\begin{array}{ll} p_1 \in \mathcal{T}^{loc}_{\varepsilon,\rho_1}(\tau), p_2 \in \mathcal{T}^{loc}_{\varepsilon,\rho_2}(\tau) & \text{and} \\ 0 < \rho_1 \le \rho_2 \le \hat{\rho} & \end{array}$$

we have

$$D(p_1) = D(p_2).$$

Because $\mathcal{X}^{loc}_{\varepsilon}(\tau)$ can be parametrized by a continuous curve, there is a continuous and bijective function $c_> : [0,1] \to \mathcal{D}^{loc}_{\varepsilon,\hat{\rho}}(\tau)$ with

$$\begin{array}{l} c_>(0) = 0 \text{ and} \\ c_>(1) = \mathcal{T}^{loc}_{\varepsilon,\hat{\rho}}(\tau) \cap \mathcal{X}^{loc}_{\varepsilon,>}(\tau) \end{array}$$

Then there are $0 < t_1 \le t_2 \le 1$ with

$$c_>(t_1) = p_1 \text{ and } c_>(t_2) = p_2.$$

As we have seen, for every $t \in [t_1, t_2]$ we get a control set $D(c_>(t)) \subset \mathbb{R}^d$ with

$$c_>(t) \in \operatorname{int} D(c_>(t)).$$

It follows that for every $t \in [t_1, t_2]$ there is an open neighborhood $V(c_>(t))$ of $c_>(t)$ with

$$V(c_>(t)) \subset D(c_>(t)).$$

Since $\{V(c_>(t)) : t \in [t_1, t_2]\}$ is an open covering of the compact set $\{c_>(t) : t \in [t_1, t_2]\}$, it follows, that there is *finite* covering

$$\{V(c_>(s_i)) : i = 1, \ldots, n\} \text{ with } t_1 \le s_0 < \ldots s_n \le t_2, n \in \mathbb{N}$$

of $\{c_>(t) : t \in [t_1, t_2]\}$. Thus for every $i \in \{1, \ldots, n\}$ there is an $j \in \{1, \ldots, n\}$ such that

$$V(c_>(s_i)) \cap V(c_>(s_j)) \neq \emptyset$$

By the maximality property of control sets it follows $D(c_>(s_i)) = D(c_>(s_j))$ and we get $D(c_>(t_1)) = D(c_>(t)) = D(c_>(t_2))$ for all $t \in [t_1, t_2]$. Therefore it follows $D(p_1) = D(p_2)$.

We will denote from now on by $D(\tau)$ the control set with $\{p \in \mathcal{X}^{loc}_{\varepsilon,>}(\tau) \cap \mathcal{T}^{loc}_{\varepsilon,\rho}(\tau) : 0 < \rho \leq \hat{\rho}\} \subset \operatorname{int} D(\tau)$.

Next we show, that for every $\tau_1, \tau_2 \in \mathbb{R}$ we have $D(\tau_1) = D(\tau_2)$. First note, that for every $\tau \in \mathbb{R}, k \in \mathbb{Z}$

$$D(\tau + 2k\Theta) = D(\tau),$$

because $\mathcal{X}^{loc}_{\varepsilon}(\tau + 2k\Theta) = \mathcal{X}^{loc}_{\varepsilon}(\tau)$. Let $p \in \mathcal{X}^{loc}_{\varepsilon,>}(\tau_2) \cap \mathcal{T}^{loc}_{\varepsilon,\rho}(\tau_2)$ for a $\rho \in (0, \hat{\rho})$. By construction we have

$$\{\varphi(t, 0, p, u^h(\tau_2 + \cdot)) : t < 0\} \subset \operatorname{int} D(\tau_2).$$

Because

$$\varphi(t, 0, p, u^h(\tau_2 + \cdot)) = \varphi(t + \tau_2, \tau_2, p, u^h) \in \mathcal{X}^{loc}_{\varepsilon}(t + \tau_2) \cap \mathcal{X}^{loc}_{\varepsilon,>}(t + \tau_2) \text{ for all } t \leq 0$$

there is a $k \in \mathbb{N}$ with

$$\begin{aligned} \varphi(\tau_1 - \tau_2 - 2k\Theta, 0, p, u^h(\tau_2 + \cdot)) &= \varphi(\tau_1 - \tau_2 - 2k\Theta + \tau_2, \tau_2, p, u^h) \\ &\in \mathcal{X}^{loc}_{\varepsilon}(\tau_1 - 2k\Theta) \cap \mathcal{X}^{loc}_{\varepsilon,>}(\tau_1 - 2k\Theta) \\ &= \mathcal{X}^{loc}_{\varepsilon}(\tau_1) \cap \mathcal{X}^{loc}_{\varepsilon,>}(\tau_1) \end{aligned}$$

and

$$\varphi(\tau_1 - \tau_2 - 2k\Theta, 0, p, u^h(\tau_2 + \cdot)) \in \mathcal{T}^{loc}_{\varepsilon,\rho'}(\tau_1)$$

for some $\rho' \in (0, \hat{\rho}]$. Thus $D(\tau_1) \cap D(\tau_2) \neq \emptyset$ and by maximality of control sets we have $D(\tau_1) = D(\tau_2) =: D_>$.

The proof for the $_<$-part of the Theorem works in the same way, as we have done here for the $_>$-part.

Finally choose the neighborhood $W \subset \mathbb{R}$ of 0, such that

$$W \cap \mathcal{D}^{loc}_{\varepsilon,\hat{\rho}}(t) = W \cap \mathcal{X}^{loc}_{\varepsilon}(t) \text{ for all } t \in \mathbb{R}. \tag{4.30}$$

Note that it is possible to choose such a neighborhood. For contradiction suppose, that there is a sequence $(t_n, p_n)_{n\in\mathbb{N}} \subset \mathcal{X}^{loc}_{\varepsilon}$ with $\|p_n\| < \frac{1}{n}$ and $(t_n, p_n) \notin \mathcal{D}^{loc}_{\varepsilon,\hat{\rho}}$. We can assume that $t_n \in [0, 2\Theta]$. Then there is a convergent subsequence which we denote again by $(t_n, p_n)_{n\in\mathbb{N}}$ and $\lim_{n\to\infty}(t_n, p_n) = (t^*, 0)$. Now $\mathcal{H}_\varepsilon(t_n, \mathcal{F}(t_n)p_n) \in X \times \{0\}$ and it follows, that $\lim_{n\to\infty} \mathcal{H}_\varepsilon(t_n, \mathcal{F}(t_n)\, p_n) = (0, 0)$. This is a contradiction, because then there must be a $n_0 \in \mathbb{N}$ with $\|\mathcal{H}(t_{n_0}, \mathcal{F}(t_{n_0})\, p_{n_0})\| < \hat{\rho}$ and it would follow, that $(t_{n_0}, p_{n_0}) \in \mathcal{D}^{loc}_{\varepsilon,\hat{\rho}}(t_{n_0})$.

Next we have to show, that the control sets $D_>, D_<$ are independent of the chosen ε. We have shown, that for $\hat{\varepsilon}$ there is an open neighborhood $\hat{W} \subset \mathbb{R}^d$ and two control sets $\hat{D}_<$ and $\hat{D}_>$ such that for every $\tau \in \mathbb{R}$ we have

$$\{\varphi(t, p_>, u^h(\tau + \cdot)) : t < 0\} \subset \operatorname{int} D_> \tag{4.31}$$

for every $p_> \in \mathcal{X}^{loc}_{\hat{\varepsilon},>}(\tau) \cap W \setminus \{x^*\}$ and

$$\{\varphi(t, p_<, u^h(\tau + \cdot)) : t < 0\} \subset \operatorname{int} D_< \tag{4.32}$$

for every $p_< \in \mathcal{X}^{loc}_{\hat{\varepsilon},<}(\tau) \cap W \setminus \{x^*\}$.

Now choose $\varepsilon \in (0, \hat{\varepsilon}]$ and the corresponding neighborhood $W \subset \mathbb{R}^d$ of 0 with the control sets $D_<$ and $D_>$ and the properties (4.25) and (4.26). Because $\hat{\varepsilon} > \varepsilon$ there is according to Proposition 6.2.11 a neighborhood $V \subset W \cap \hat{W}$ such that for all $t \in \mathbb{R}$ we have

$$\mathcal{X}^{loc}_{\varepsilon}(t) \cap V = \mathcal{X}^{loc}_{\hat{\varepsilon}}(t) \cap V. \tag{4.33}$$

By (4.31) there exists a $p_> \in \operatorname{int} \hat{D}_>$ with $p_> \in \mathcal{X}^{loc}_{\hat{\varepsilon},>}(t) \cap V$ for some $t \in \mathbb{R}$. But because of (4.33) it follows, that $p_> \in \mathcal{X}^{loc}_{\varepsilon,>}(t) \cap V$ and therefore $D_> = \hat{D}_>$. Furthermore there exists a $p_< \in \operatorname{int} \hat{D}_<$ with $p_< \in \mathcal{X}^{loc}_{\hat{\varepsilon},<}(t) \cap V$ for some $t \in \mathbb{R}$. But because of (4.33) it follows, that $p_< \in \mathcal{X}^{loc}_{\varepsilon,<}(t) \cap V$ and therefore $D_< = \hat{D}_<$. ■

We want to compare this theorem with the General Existence Theorem 3.7.1, where we assume for simplicity, that the condition (1)- (5) of Theorem 4.1.12 are fulfilled.

The first difference is, that in the new Theorem 4.1.12 the control sets $D_<$ and $D_>$ do *not* depend on the chosen $\tau \in \mathbb{R}$. In Theorem 3.7.1 we got for every $\tau \in \mathbb{R}$ a control set $D(\tau)$. In Figure 4.5 we have drawn the control set $D_>(\tau_1)$ and $D_>(\tau_2)$ for $\tau_1 \neq \tau_2$. Theorem 3.7.1 does not guarantee, that $D_>(\tau_1)$ and $D_>(\tau_2)$ coincide, cf. Figure 4.5 (a). But under the slightly stronger assumptions of Theorem 4.1.12, we get that $D_>(\tau_1) = D_>(\tau_2)$, cf. Figure 4.5 (b).

Furthermore, Theorem 3.7.1 only states, that there is *one* control set, but not if it is $D_>$ or $D_<$. With Theorem 4.1.12 it is guaranteed, that both control sets exist. This was possible because we used the topological structure of the unstable fibre bundle, which is here onedimensional.

Then the characterization via the trajectories $\varphi(t, 0, p^*, u^*)$ can be different. Both theorems show, that there can be a $p \in \mathcal{X}^{loc}_{\varepsilon}(\tau)$ such that

$$\begin{aligned} p &\in \operatorname{int} D \text{ and} \\ \varphi(t, 0, p, u^s) &\subset \operatorname{int} D. \end{aligned}$$

But in the onedimensional case, an additional situation may occur, which is not specified by the general Theorem 3.7.1. If $\varphi(t, 0, p, u^s) \in \mathcal{X}^{loc}_{\varepsilon,>}(\tau)$, then the result from Theorem

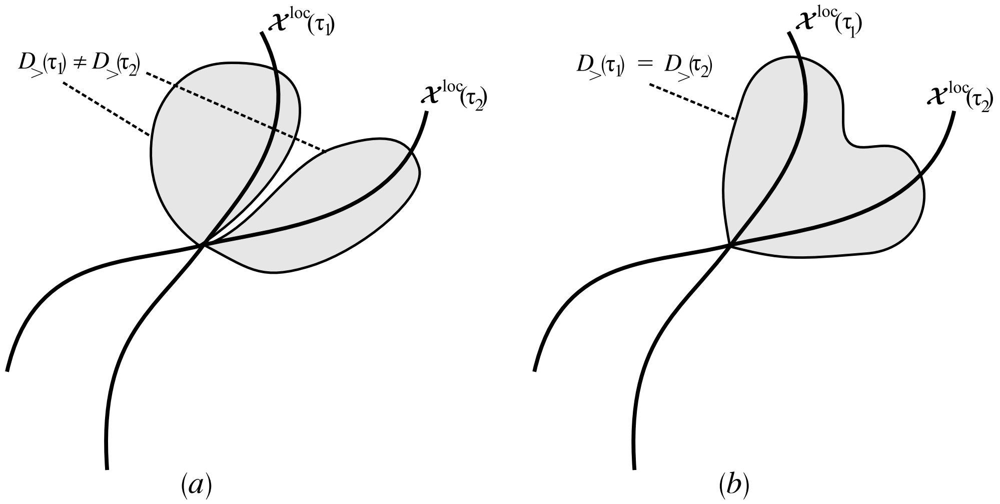

Figure 4.5: Comparison of the two construction algorithms.

4.1.12 is shown in Figure 4.6 (a), which is similar to Figure 3.5 which we got from the general Theorem 3.7.1. If there is $\xi > 0$ such that $\varphi(\xi, 0, p, u^s) \in \mathcal{Y}_\varepsilon^{loc}(\tau)$, then it is possible, that there is a $\zeta \geq \xi$ such that with the control function v^* also a part of the stable fibre bundle $\mathcal{Y}_\varepsilon^{loc}(\tau)$ lies in the interior of the control set. This is illustrated in Figure 4.6 (b).

Furthermore, Theorem 4.1.12 guarantees, that the control set $D_>$ and $D_<$ are independent of the chosen local stable fibre bundle, i.e. independent of ε.

At the moment it is still unclear, when $D_<$ and $D_>$ coincide. In the next chapter, we will give some criterions under which this situation may occur.

4.2 Two Hyperbolic Systems

If one can find for a nonlinear control system with singular point a periodic control function $u_1^h \in \mathcal{U}$ such that the corresponding Lyapunov exponents fulfill

$$\lambda_{1,1}^h > 0 > \lambda_{1,2}^h \geq \ldots \geq \lambda_{1,d}^h$$

then one can often expect to find another periodic control function $u_2^h \in \mathcal{U}$ with the same property:

$$\lambda_{2,1}^h > 0 > \lambda_{2,2}^h \geq \ldots \geq \lambda_{2,d}^h.$$

Suppose for instance, that $\mathcal{U} \times \mathbb{R}^d = \mathcal{V}_1 \oplus \mathcal{V}_2$ for two invariant subbundles $\mathcal{V}_1$ and $\mathcal{V}_2$ which we get from Theorem 1.2.7 such that

$$0 \in \operatorname{int} \Sigma_{Ly}(\mathcal{V}_1) \text{ and } \Sigma_{Ly}(\mathcal{V}_2) \subset \mathbb{R}^-$$

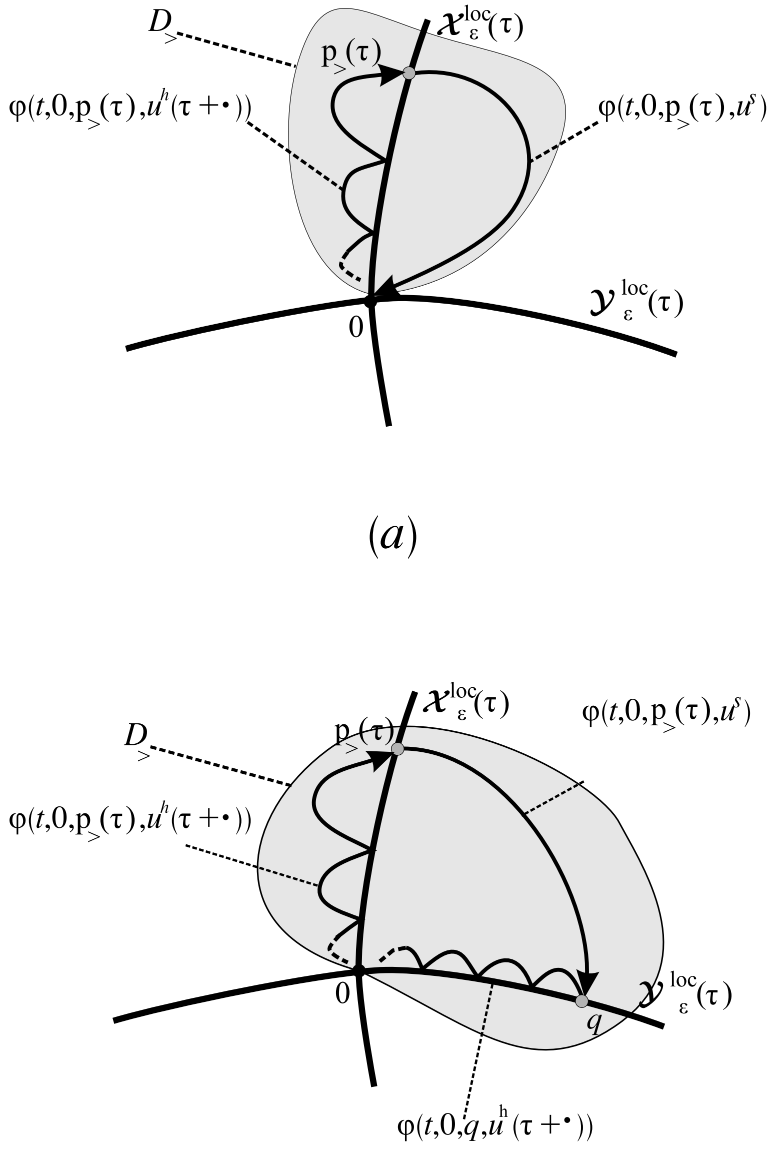

Figure 4.6: Two possible shapes of $D_>$.

with

$$\dim \mathcal{V}_1 = 1 \text{ and } \Sigma_{Ly} = \text{cl}\, \Sigma_{Fl} \tag{4.34}$$

(cf. Theorem 1.2.11). In this case there are infinitely many of such control functions: For $\lambda_1 \in \Sigma_{Ly}(\mathcal{V}_1)$ with $\lambda_1 > 0$ and $\lambda_i \in \Sigma_{Ly}(\mathcal{V}_2), i = 2, \ldots, d$ and every $\sigma > 0$ such that $\lambda_1 - \sigma > 0$ there is by (4.34) a periodic control function $u^h \in \mathcal{U}$ such that the corresponding linearized system has the Lyapunov exponents $\lambda_i^h \in \mathbb{R}$ with $|\lambda_i - \lambda_i^h| < \sigma$.

Now for two control functions u_1^h and u_2^h with this property we get (if the corresponding assumptions are fulfilled) by Theorem 4.1.12 four control sets $D_>(u_i^h)$ and $D_<(u_i^h)$ for $i = 1, 2$. In this section we will now analyze, which of these control sets coincide. There is no criterion, which indicates if two of these control sets do *not* coincide. However, we will show, which of these control sets lie in the domain of attraction of another control set.

The basic idea is to construct a control function u by switching between u_1^h and u_2^h and a stable control function u^s. Then it depends on the position of the unstable fibre bundles of the two control function u_1^h and u_2^h which ones of the control sets coincide.

4.2.1 Local Classification

We consider the nonlinear control system on $\mathbb{R}^d$

$$\begin{array}{l} \dot{x} = f_0(x) + \sum_{i=1}^m u_i(t)\, f_i(x) \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a } t \in \mathbb{R}, \text{locally integrable}\} \end{array} \tag{4.35}$$

where U is a compact and convex subset of $\mathbb{R}^m$ and $f_0, \ldots, f_m$ are C^2 vector fields on $\mathbb{R}^d$. Furthermore, suppose that for all $(u, x) \in \mathcal{U} \times \mathbb{R}^d$ the equation (4.35) has a unique solution $\varphi(t, \tau, x, u), t, \tau \in \mathbb{R}$, with $\varphi(\tau, \tau, x, u) = x$.

We suppose, that the nonlinear system (4.4) has the origin $0 \in \mathbb{R}^d$ as singular point.

Notation 4.2.1 *From now on, we supply all the objects, which belong to the control function u_1^h (for example the stable or unstable fibre bundle) with subscript $_1$. Then the local unstable fibre bundle which belongs to u_1^h is denoted by $\mathcal{X}_{1,\varepsilon}^{loc}$ and the fibre at time $t \in \mathbb{R}$ with $\mathcal{X}_{1,\varepsilon}^{loc}(t)$. In the same way we supply all the objects which belong to u_2^h with the subscript $_2$.*

First of all, suppose that the following assumptions are fulfilled.

Condition 4.2.2 *We assume, that there are three periodic control function $u_1^h, u_2^h, u^s \in \mathcal{U}$ with the following properties:*

(a) The control function u_1^h has period $\Theta_1 \geq 0$ and the Lyapunov exponents $\lambda_{1,1}^h, \ldots, \lambda_{1,d}^h$ of the corresponding linear system $\dot{x} = A_0 x + \sum_{i=1}^m u_{1,i}^h(t) A_i x$ have the property

$$\lambda_{1,1}^h > 0 > \lambda_{1,2}^h \geq \ldots \geq \lambda_{1,d}^h.$$

(b) *The control function* u_2^h *has period* $\Theta_2 \geq 0$ *and the Lyapunov exponents* $\lambda_{2,1}^h, \ldots, \lambda_{2,d}^h$ *of the corresponding linear system* $\dot{x} = A_0 x + \sum_{i=1}^m u_{1,i}^h(t) A_i x$ *have the property*

$$\lambda_{2,1}^h > 0 > \lambda_{2,2}^h \geq \ldots \geq \lambda_{2,d}^h.$$

(c) *The Lyapunov exponents* $\lambda_1^s, \ldots, \lambda_d^s$ *of the linear system* $\dot{x} = A_0 x + \sum_{i=1}^m u_i^s(t) A_i x$ *have the property*

$$0 > \lambda_1^s \geq \lambda_2^s \geq \ldots \geq \lambda_d^s.$$

For the definition of $\hat{\varepsilon}_1, \hat{\varepsilon}_2$, we refer to Notation 3.2.10 where we consider the systems

$$\dot{x} = f_0(x) + \sum_{\iota=1}^m u_{1,i}^h(t) f_i(x) \text{ and}$$
$$\dot{x} = f_0(x) + \sum_{i=1}^m u_{2,i}^h(t) f_i(x)$$

Define

$$\hat{\varepsilon} := \min\{\hat{\varepsilon}_1, \hat{\varepsilon}_2\}. \tag{4.36}$$

For every $\varepsilon \in (0, \hat{\varepsilon}]$ we get for u_i^h the local unstable and the local stable fibre bundles $\mathcal{X}_{i,\varepsilon}^{loc}$ and $\mathcal{Y}_{i,\varepsilon}^{loc}$.

We will make a similar construction as in Section 4.1, thus we have to impose the following condition.

Condition 4.2.3 *The nonlinear control system (4.35) is locally accessible on* $\mathbb{R}^d \setminus \{0\}$ *and there is a neighborhood* $V \subset \mathbb{R}^d$ *such that for all* $\varepsilon \in (0, \hat{\varepsilon}], t \in \mathbb{R}$ *and all* $x \in \mathcal{Y}_{j,\varepsilon}^{loc}(t) \cap V \setminus \{0\}$ *the pairs* $(u_j^h(t + \cdot), x)$ *are strong inner pairs for* $j = 1, 2$.

The next proposition describes, which points of the local unstable fibre bundle $\mathcal{X}_{1,\varepsilon}^{loc}(\tau_1)$ can be steered to $\mathcal{X}_{2,\varepsilon}^{loc}(\tau_2)$ and, vice versa, which points of $\mathcal{X}_{2,\varepsilon}^{loc}(\tau_2)$ can be steered to $\mathcal{X}_{1,\varepsilon}^{loc}(\tau_1)$.

Proposition 4.2.4 *Suppose, that the nonlinear control system (4.35) satisfies Condition 4.2.2 and 4.2.3. Let* $\varepsilon \in (0, \hat{\varepsilon}], \tau_1, \tau_2 \in \mathbb{R}$ *and* $\sigma > 0$. *Then there exists a neighborhood* $N \subset \mathbb{R}^d$ *of* 0 *such that for all*

$$p_{j,>} \in \mathcal{X}_{j,\varepsilon}^{loc}(\tau_j) \cap \mathcal{X}_{j,\varepsilon,>}^{loc}(\tau_j) \cap N,$$
$$p_{j,<} \in \mathcal{X}_{j,\varepsilon}^{loc}(\tau_j) \cap \mathcal{X}_{j,\varepsilon,<}^{loc}(\tau_j) \cap N,$$

for $j = 1, 2$ *with*

$$p_{1,>}, p_{1,<} \notin \mathcal{Y}_{2,\varepsilon}^{loc}(\tau_2), \tag{4.37}$$
$$p_{2,>}, p_{2,<} \notin \mathcal{Y}_{2,\varepsilon}^{loc}(\tau_2), \tag{4.38}$$

we get: For each of the following cases, there exists a control function $u \in \mathcal{U}$ *and a time* $T > 0$ *such that if*

(a) $p_{1,>} \in \mathcal{X}^{loc}_{2,\varepsilon,>}(\tau_2)$ *then* $\varphi(T,0,p_{1,>},u) \in B_\sigma(p_{2,>})$.

(b) $p_{1,>} \in \mathcal{X}^{loc}_{2,\varepsilon,<}(\tau_2)$ *then* $\varphi(T,0,p_{1,>},u) \in B_\sigma(p_{2,<})$.

(c) $p_{1,<} \in \mathcal{X}^{loc}_{2,\varepsilon,>}(\tau_2)$ *then* $\varphi(T,0,p_{1,<},u) \in B_\sigma(p_{2,>})$.

(d) $p_{1,<} \in \mathcal{X}^{loc}_{2,\varepsilon,<}(\tau_2)$ *then* $\varphi(T,0,p_{1,<},u) \in B_\sigma(p_{2,<})$.

(e) $p_{2,>} \in \mathcal{X}^{loc}_{1,\varepsilon,>}(\tau_1)$ *then* $\varphi(T,0,p_{2,>},u) \in B_\sigma(p_{1,>})$.

(f) $p_{2,>} \in \mathcal{X}^{loc}_{1,\varepsilon,<}(\tau_1)$ *then* $\varphi(T,0,p_{2,>},u) \in B_\sigma(p_{1,<})$.

(g) $p_{2,<} \in \mathcal{X}^{loc}_{1,\varepsilon,>}(\tau_1)$ *then* $\varphi(T,0,p_{2,<},u) \in B_\sigma(p_{1,>})$.

(h) $p_{2,<} \in \mathcal{X}^{loc}_{1,\varepsilon,<}(\tau_1)$ *then* $\varphi(T,0,p_{2,<},u) \in B_\sigma(p_{1,<})$.

Proof. For some fixed $\varepsilon \in (0,\hat{\varepsilon}]$ we get a neighborhood $W_i(\varepsilon) \subset \mathbb{R}^d$ of 0 such that for all $p \in \mathcal{X}^{loc}_{i,\varepsilon}(\tau) \cap W_i(\varepsilon)$ and all $q \in \mathcal{Y}^{loc}_{i,\varepsilon}(\tau) \cap W_i(\varepsilon)$ we have for $i = 1,2$

$$\begin{aligned}\varphi(t,\tau_i,p,u_i^h) &= \mathcal{F}_i^{-1}(t)\psi(t,\tau_i,\mathcal{F}_i(t)p,u_i^h) = \mathcal{F}_i^{-1}(t)\mu_\varepsilon(t,\tau_i,\mathcal{F}(t)p,u_i^h) \text{ for all } t \le \tau_i,\\ \varphi(t,\tau_i,q,u_i^h) &= \mathcal{F}_i^{-1}(t)\psi(t,\tau_i,\mathcal{F}_i(t)q,u_i^h) = \mathcal{F}_i^{-1}(t)\mu_\varepsilon(t,\tau_i,\mathcal{F}(t)q,u_i^h) \text{ for all } t \ge \tau_i.\end{aligned} \tag{4.39}$$

Choose $V^s \subset W_1(\varepsilon) \cap W_2(\varepsilon)$ and by Lemma 6.2.10 there exists an open neighborhood $W^s \subset \mathbb{R}^d$ of 0, such that for all $p \in W^s$ we have

$$\begin{aligned}\varphi(t,0,p,u^s) &\in V^s \quad \text{for all } t \ge 0 \text{ and}\\ \lim_{t\to\infty}\varphi(t,s,p,u^s) &= 0.\end{aligned} \tag{4.40}$$

By Lemma 3.2.8 there is a $\hat{\rho}_j := \hat{\rho}_j(\varepsilon, W^s) \in (0,\rho_i^*(\varepsilon))$ such that

$$\mathcal{D}^{loc}_{j,\varepsilon,\rho}(t) \subset W^s \text{ and for all } \rho \in (0,\hat{\rho}_j), t \in \mathbb{R}, j = 1,2. \tag{4.41}$$

Thus if we define $\hat{\rho} := \min\{\hat{\rho}_1,\hat{\rho}_2\}$ we have for all $\rho \in (0,\hat{\rho}]$

$$\mathcal{T}^{loc}_{j,\varepsilon,\rho}(t) \subset W^s \subset V^s \subset W_j(\varepsilon) \text{ for all } t \in \mathbb{R}, j = 1,2. \tag{4.42}$$

Finally choose the neighborhood $N \subset \mathbb{R}$ of 0, such that

$$N \cap \mathcal{D}^{loc}_{j,\varepsilon,\hat{\rho}}(t) = N \cap \mathcal{X}^{loc}_{j,\varepsilon}(t) \text{ for all } t \in \mathbb{R}, j = 1,2. \tag{4.43}$$

Note that it is possible to choose such a neighborhood. Otherwise suppose, that there would be a sequence $(t_n,p_n)_{n\in\mathbb{N}} \subset \mathcal{X}^{loc}_{j,\varepsilon}$ with $\|p_n\| < \frac{1}{n}$ and $(t_n,p_n) \notin \mathcal{D}^{loc}_{j,\varepsilon,\hat{\rho}}$. We can assume that $t_n \in [0,2\Theta_j]$. Then there is a convergent subsequence which we denote again by $(t_n,p_n)_{n\in\mathbb{N}}$ and $\lim_{n\to\infty}(t_n,p_n) = (t^*,0)$. Now $\mathcal{H}_{j,\varepsilon}(t_n,\mathcal{F}_j(t_n)\,p_n) \in X_j \times \{0\}$ and it follows, that $\lim_{n\to\infty}\mathcal{H}_{j,\varepsilon}(t_n,\mathcal{F}_j(t_n)\,p_n) = (0,0)$. But this is a contradiction, because then there must be an $n_0 \in \mathbb{N}$ with $\|\mathcal{H}_j(t_{n_0},\mathcal{F}_j(t_{n_0})\,p_{n_0})\| < \hat{\rho}$ and it would follow, that $(t_{n_0},p_{n_0}) \in \mathcal{D}^{loc}_{j,\varepsilon,\hat{\rho}}(t_{n_0})$.

For illustration we show (a), the other assumptions follow in the same way. For $\tau_1, \tau_2 \in \mathbb{R}$ choose

$$p_{j,>} \in \mathcal{X}^{loc}_{j,\varepsilon}(\tau_j) \cap \mathcal{X}^{loc}_{j,\varepsilon,>} \cap N,$$
$$p_{j,<} \in \mathcal{X}^{loc}_{j,\varepsilon}(\tau_j) \cap \mathcal{X}^{loc}_{j,\varepsilon,<} \cap N,$$

such that (4.37) and (4.38) is fulfilled. Choose $\sigma > 0$ eventually smaller such that

$$B_\sigma(\mathcal{T}^{loc}_{1,\varepsilon,\hat{\rho}}(\tau_1)) \cap \mathcal{Y}^{loc}_{2,\varepsilon}(\tau_2) = \emptyset, \tag{4.44}$$
$$B_\sigma(\mathcal{T}^{loc}_{2,\varepsilon,\hat{\rho}}(\tau_2)) \cap \mathcal{Y}^{loc}_{1,\varepsilon}(\tau_1) = \emptyset, \tag{4.45}$$
$$B_\sigma(p_{j,>}) \cap B_\sigma(p_{j,<}) = \emptyset, \tag{4.46}$$
$$B_\sigma(\mathcal{D}^{loc}_{j,\varepsilon,\hat{\rho}}) \subset W^s$$

By Proposition 3.3.3 there is a $\delta > 0$ such that we have:

For every $\tau \in \mathbb{R}, q \in \mathcal{T}^{loc}_{2,\varepsilon,\hat{\rho}}(\tau)$ there is a $K \in \mathbb{N}$ such that for every $k \geq K, k \in \mathbb{N}$ we have

$$\varphi(-2k\Theta_2 + \tau_2, \tau_2, q, u_2^h) \in B_\delta(0). \tag{4.47}$$

Furthermore, if $p \in B_\delta(0)$ and $\mathcal{P}^{loc}_{2,\varepsilon}(\tau_2, p) = \varphi(-2k\Theta_2 + \tau_2, \tau_2, q, u_2^h)$, then

$$\varphi(t, \tau_2, p, u_2^h) \in B_\sigma(\mathcal{X}^{loc}_{2,\varepsilon}(t)) \text{ for every } t \in [\tau, \tau + 2k\Theta_2] \text{ and} \tag{4.48}$$
$$\varphi(2k\Theta_2 + \tau_2, \tau_2, p, u_2^h) \in B_\sigma(\mathcal{T}^{loc}_{2,\varepsilon,\hat{\rho}}(\tau_2)) = B_\sigma(p_{2,>}(\tau_2)) \cup B_\sigma(p_{2,>}(\tau_2)). \tag{4.49}$$

Now there are two cases. Either

$$\varphi(t, 0, p_{1,>}, u^s) \in \mathcal{X}^{loc}_{2,\varepsilon,>}(\tau_2) \text{ for all } t > 0$$

or there is a time $\xi > 0$ with

$$\varphi(\xi, 0, p_{1,>}, u^s) \in \mathcal{Y}^{loc}_{2,\varepsilon}(\tau_2) \text{ and}$$
$$\varphi(t, 0, p_{1,>}, u^s) \in \mathcal{X}^{loc}_{2,\varepsilon,>}(\tau_2) \text{ for all } 0 < t < \xi.$$

If the *first case* is fulfilled, because $p_{1,>}(\tau_1) \in W^s$ there exists a time $t_0 > 0$ with

$$\varphi(t, 0, p_{1,>}, u^s) \in B_\delta(0) \ \cap \mathcal{X}^{loc}_{\varepsilon,>}(\tau) \text{ for all } t > t_0.$$

Now because the projection mapping $\mathcal{P}^{loc}_{2,\varepsilon}$ is continuous with $\mathcal{P}^{loc}_{2,\varepsilon}(\tau_2, 0) = 0$ and because $\lim_{t\to\infty} \varphi(t, 0, p_{1,>}, u^s) = 0$, we get $\lim_{t\to\infty} \mathcal{P}^{loc}_{2,\varepsilon}(\tau_2, \varphi(t, 0, p_{1,>}, u^s)) = 0$. Thus by Proposition 3.2.11 there is a $k \in \mathbb{N}$ with $k > K$ and $T_0 > t_0$ with

$$\mathcal{P}^{loc}_{2,\varepsilon}(\tau_2, \varphi(T_0, 0, p_{1,>}, u^s)) = \varphi(-2k\Theta_2 + \tau_2, \tau_2, q, u_2^h)$$

for a $q \in \mathcal{T}^{loc}_{2,\varepsilon,\rho}(\tau_2)$. By relation (4.49) it follows, that

$$\varphi(2k\Theta_2 + \tau_2, \tau_2, p, u_2^h) \in B_\sigma(\mathcal{T}^{loc}_{2,\varepsilon,\rho}(\tau_2)) = B_\sigma(p_{2,>}) \cup B_\sigma(p_{2,>}).$$

But because $\varphi(T_0, 0, p_{1,>}, u^s) \in \mathcal{X}^{loc}_{2,\varepsilon,>}(\tau_2)$ we have $\varphi(2k\Theta_2 + \tau_2, \tau_2, p, u^h_2) \in B_\sigma(p_{2,>})$. Thus by defining

$$u_1(t) := \begin{cases} 0 & \text{for } t \in (-\infty, 0), \\ u^s(t) & \text{for } t \in [0, T_0), \\ u^h_2(t + \tau_2 - T_0) & \text{for } t \in [T_0, \infty). \end{cases}$$

and

$$T := T_0 + 2k\Theta_2$$

the assumption follows.

In the *second case* we define

$$q_0 := \varphi(\xi, 0, p_{1,>}, u^s) \in V^s.$$

Because $q_0 \in \mathcal{Y}^{loc}_{2,\varepsilon}(\tau_2)$ there is a time $l \in \mathbb{N}$ such that

$$q_1 := \varphi(2l\Theta_2 + \tau_2, \tau_2, q_0, u^h_2) = \varphi(2l\Theta_2, 0, q_0, u^h(\tau_2 + \cdot) \in B_\sigma(0).$$

By periodicity of $\mathcal{Y}^{loc}_{2,\varepsilon}(\cdot)$ we have $q_1 \in B_\sigma(0) \cap \mathcal{Y}^{loc}_{2,\varepsilon}(2l\Theta_2 + \tau_2) = B_\sigma(0) \cap \mathcal{Y}^{loc}_{2,\varepsilon}(\tau_2)$. By assumption (u^h_2, q_2) is a strong inner pair, thus there exists a time $\hat{T}_1 > 0$ and a control function $v \in \mathcal{U}$ such that

$$\begin{array}{lcll} \varphi(\hat{T}_1, 0, q_1, v) & \in & B_\sigma(0) \cap \mathcal{X}^{loc}_{2,\varepsilon,>}(\tau_2) & \text{and} \\ \varphi(t, 0, q_1, v) & \in & B_\sigma(0) & \text{for all } t \in [0, \hat{T}_1]. \end{array}$$

Then there is an interval $[a, b] \subset [0, \hat{T}_1]$ with

$$\begin{array}{ll} \varphi(a, 0, q_1, v) \in \mathcal{Y}^{loc}_{2,\varepsilon}(\tau_2) & \text{and} \\ \varphi(t, 0, q_1, v) \in \mathcal{X}^{loc}_{2,\varepsilon,>}(\tau_2) & \text{for all } t \in (a, b]. \end{array}$$

Because $\varphi(a, 0, q_1, v) \in \mathcal{Y}^{loc}_{2,\varepsilon}(\tau_2)$ it follows $\mathcal{P}^{loc}_{2,\varepsilon}(\tau_2, \varphi(a, 0, q_1, V)) = 0$, and because $\varphi(t, 0, q_1, v) \notin \mathcal{Y}^{loc}(\tau)$ we get $\mathcal{P}^{loc}_{2,\varepsilon}(\tau_2, \varphi(t, 0, q_1, v)) \in \mathcal{X}^{loc}_{2,\varepsilon}(\tau_2) \cap \mathcal{X}^{loc}_{2,\varepsilon,>}(\tau_2) \setminus \{0\}$ for all $t \in (a, b]$. By Proposition 3.2.11 there is a $k \in \mathbb{N}$ with $k > K$ and a $T_1 \in (a, b]$ with

$$\mathcal{P}^{loc}_{2,\varepsilon}(\tau_2, \varphi(T_1, 0, q_1, v)) = \varphi(-2k\Theta_2 + \tau_2, \tau_2, p_{2,>}, u^h_2).$$

If we define $q_2 := \varphi(T_1, 0, q_1, v)$, then with relation (4.49) it follows

$$\varphi(2k\Theta_2 + \tau_2, \tau_2, q_2, u^h_2) \in B_\sigma(p_{2,>}).$$

Thus by defining

$$T := \xi + 2l\Theta_2 + T_1 + 2k\Theta_2$$

and

$$u_2(t) := \begin{cases} 0 & \text{for } t \in (-\infty, 0), \\ u^s(t) & \text{for } t \in [0, \xi), \\ u_2^h(t + \tau_2 - \xi) & \text{for } t \in [\xi, 2l\Theta_2), \\ v(t - T_0) & \text{for } t \in [2l\Theta_2, T_1), \\ u_2^h(t + \tau - T_1) & \text{for } t \in [T_1, \infty) \end{cases}$$

the assumptions follows. ■

We are now in the position to give a local classification of the control sets near the singular point.

Theorem 4.2.5 *Consider the nonlinear control system*

$$\begin{array}{l} \dot{x} = f_0(x) + \sum_{i=1}^m u_i(t) f_i(x) \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \textit{ for a.a. } t \in \mathbb{R}, \textit{locally integrable}\} \end{array} \tag{4.50}$$

where U is a compact and convex subset of $\mathbb{R}^m$ and $f_0, \ldots, f_m$ are C^2 vector fields on $\mathbb{R}$. Suppose that for all $(u,x) \in \mathcal{U} \times \mathbb{R}^d$ the equation (4.50) has a unique solution $\varphi(t,\tau,x,u), t, \tau \in \mathbb{R}$, with $\varphi(\tau,\tau,x,u) = x$. Assume, that following properties are fulfilled.

(1) The nonlinear control system (4.50) has one singular point $x^* = 0 \in \mathbb{R}^d$ and is Lie-determined such that $\mathbb{R}^d \setminus \{0\}$ and $\{0\}$ are maximal integral manifolds.

(2) There are three periodic control functions u_1^h, u_2^h and $u^s \in \mathcal{U}$ such that the associated Lyapunov exponents of the linearized systems have the following properties

$$\begin{array}{rcl} & 0 & > \lambda_1^s \geq \ldots \geq \lambda_d^s, \\ \lambda_{1,1}^h > & 0 & > \lambda_{1,2}^h \geq \ldots \geq \lambda_{1,d}^h, \\ \lambda_{2,1}^h > & 0 & > \lambda_{2,2}^h \geq \ldots \geq \lambda_{2,d}^h. \end{array}$$

Then define $\hat{\varepsilon}$ by (4.36), choose $\varepsilon \in (0, \hat{\varepsilon}]$ and denote by $\mathcal{X}_{j,\varepsilon}^{loc}, \mathcal{Y}_{j,\varepsilon}^{loc}$ the corresponding local unstable and stable fibre bundle of the differential equations

$$\dot{x} = f_0(x) + \sum_{i=1}^m u_{j,i}^h(t) f_i(x)$$

corresponding to $u_j^h, j = 1, 2$.

Moreover, suppose that there is a neighborhood $V \subset \mathbb{R}^d$ of 0 such that

(3) for every $t \in \mathbb{R}$ and every $x \in \mathcal{X}_{j,\varepsilon}^{loc}(t) \cap V \setminus \{0\}$ the pair $(u_j^h(t + \cdot), x)$ is a strong inner pair for $j = 1, 2$.

(4) for every $t \in \mathbb{R}$ and every $x \in \mathcal{Y}_{j,\varepsilon}^{loc}(t) \cap V \setminus \{0\}$ the pair $(u_j^h(t + \cdot), x)$ is a strong inner pair for $j = 1, 2$.

(5) for every $t \in \mathbb{R}$ *and every* $x \in V \setminus \{0\}$ *the pair* $(u^s(t+\cdot), x)$ *is a strong inner pair.*

Then there exists a neighborhood $N \subset \mathbb{R}^d$ *and four control sets* $D_{j,>}, D_{j,<} \subset \mathbb{R}^d$, $j = 1, 2$ *with nonvoid interior such that for every* $\tau \in \mathbb{R}$ *we have*

$$\{\varphi(t, p_{>,j}, u_j^h(\tau_j + \cdot)) : t \leq 0\} \subset \operatorname{int} D_{j,>} \tag{4.51}$$

$$\{\varphi(t, p_{<,j}, u_j^h(\tau_j + \cdot)) : t \leq 0\} \subset \operatorname{int} D_{j,<} \tag{4.52}$$

for every $p_{>,j} \in \mathcal{X}_{j,\varepsilon}^{loc}(\tau_j)$ *with* $p_{>,j} \in \mathcal{X}_{j,\varepsilon,>}^{loc}(\tau_j) \cap N \setminus \{0\}$ *and every* $p_{<,j} \in \mathcal{X}_{j,\varepsilon}^{loc}(\tau_j)$ *with* $p_{<,j} \in \mathcal{X}_{j,\varepsilon,<}^{loc}(\tau_j) \cap N \setminus \{0\}$ *for* $j = 1, 2$. *In particular, we have*

$$x^* \in \operatorname{cl} D_{1,>} \cap \operatorname{cl} D_{1,<} \cap \operatorname{cl} D_{2,>} \cap \operatorname{cl} D_{2,<}. \tag{4.53}$$

Furthermore let $\tau_1, \tau_2 \in \mathbb{R}$. *The interrelation of the control sets is summarized in the table of Figure 4.7. For an illustration of the table consider Figure 4.8 and Figure 4.9. The table has to bead read in the following way:*

If we look at the first column, the "$>$*" indicates, that we assume, that there is a* $p_{1,>} \in \mathcal{X}_{1,\varepsilon}^{loc}(\tau_1)$ *with* $p_{1,>} \in \mathcal{X}_{1,\varepsilon,>}^{loc}(\tau_1) \cap N \setminus \{0\}$ *and*

$$p_{1,>} \in \mathcal{X}_{2,\varepsilon,>}^{loc}(\tau_2).$$

If there is a "$<$*" in the first column, we assume that*

$$p_{1,>} \in \mathcal{X}_{2,\varepsilon,<}^{loc}(\tau_2).$$

The notations for the other columns are similar: The second column indicates, that we assume, that there is a $p_{2,>} \in \mathcal{X}_{2,\varepsilon}^{loc}(\tau_2)$ *with* $p_{2,>} \in \mathcal{X}_{2,\varepsilon,>}^{loc}(\tau_2) \cap N \setminus \{0\}$ *and*

$$p_{2,>} \in \mathcal{X}_{1,\varepsilon,>}^{loc}(\tau_1) \text{ or } p_{2,>} \in \mathcal{X}_{1,\varepsilon,<}^{loc}(\tau_1)$$

depending on the entry "$>$*" or "*$<$*". The third columns indicates, that we assume, that there is a* $p_{1,<} \in \mathcal{X}_{1,\varepsilon}^{loc}(\tau_1)$ *with* $p_{1,<} \in \mathcal{X}_{1,<}^{loc}(\tau_1) \cap N \setminus \{0\}$ *and*

$$p_{1,<} \in \mathcal{X}_{2,\varepsilon,<}^{loc}(\tau_2) \text{ or } p_{1,<} \in \mathcal{X}_{2,\varepsilon,>}^{loc}(\tau_2)$$

depending on the entry "$>$*" or "*$<$*". The fourth columns indicates, that we assume, that here is a* $p_{2,<} \in \mathcal{X}_{2,\varepsilon}^{loc}(\tau_2)$ *with* $p_{2,<} \in \mathcal{X}_{2,\varepsilon,<}^{loc}(\tau_2) \cap N \setminus \{0\}$ *and*

$$p_{2,<} \in \mathcal{X}_{1,\varepsilon,>}^{loc}(\tau_1) \text{ or } p_{2,<} \in \mathcal{X}_{1,\varepsilon,<}^{loc}(\tau_1)$$

depending on the entry "$>$*" or "*$<$*". Then the last two columns shows, which of the control sets coincide, and which control set lies in the domain of attraction of another control set.*

	$p_{1,>}(\tau_1)$	$p_{2,>}(\tau_2)$	$p_{1,<}(\tau_1)$	$p_{2,<}(\tau_2)$	Control sets	Attraction
1	$>$	$>$	$<$	$<$	$D_{1,>} = D_{2,>}, D_{1,<} = D_{2,<}$	
2	$>$	$>$	$<$	$>$	$D_{1,>} = D_{2,>}$	$D_{2,<} \subset \mathbf{A}(D_{1,>}), D_{1,<} \subset \mathbf{A}(D_{2,<})$
3	$>$	$>$	$>$	$<$	$D_{1,>} = D_{2,>}$	$D_{1,<} \subset \mathbf{A}(D_{1,>}), D_{2,<} \subset \mathbf{A}(D_{1,<})$
4	$>$	$>$	$>$	$>$	$D_{1,>} = D_{2,>}$	$D_{1,<} \subset \mathbf{A}(D_{1,>}), D_{2,<} \subset \mathbf{A}(D_{1,>})$
5	$>$	$<$	$<$	$<$	$D_{1,<} = D_{2,<}$	$D_{1,>} \subset \mathbf{A}(D_{1,<}), D_{2,>} \subset \mathbf{A}(D_{1,<})$
6	$>$	$<$	$<$	$>$	$D_{1,>} = D_{2,>} = D_{1,<} = D_{1,<}$	
7	$>$	$<$	$>$	$<$	$D_{1,<} = D_{2,>}$	$D_{1,>} \subset \mathbf{A}(D_{1,<}), D_{2,<} \subset \mathbf{A}(D_{1,<})$
8	$>$	$<$	$>$	$>$	$D_{1,<} = D_{2,>}$	$D_{1,>} \subset \mathbf{A}(D_{1,<}), D_{2,<} \subset \mathbf{A}(D_{1,<})$
9	$<$	$>$	$>$	$>$	$D_{1,>} = D_{2,<}$	$D_{1,<} \subset \mathbf{A}(D_{2,>}), D_{2,>} \subset \mathbf{A}(D_{1,>})$
10	$<$	$>$	$>$	$<$	$D_{1,>} = D_{2,>} = D_{1,<} = D_{1,<}$	
11	$<$	$>$	$<$	$>$	$D_{1,>} = D_{2,<}$	$D_{1,<} \subset \mathbf{A}(D_{1,>}), D_{2,>} \subset \mathbf{A}(D_{1,>})$
12	$<$	$>$	$<$	$<$	$D_{1,<} = D_{2,<}$	$D_{2,>} \subset \mathbf{A}(D_{1,>}), D_{1,>} \subset \mathbf{A}(D_{1,<})$
13	$<$	$<$	$>$	$>$	$D_{1,>} = D_{2,<}, D_{1,<} = D_{2,>}$	
14	$<$	$<$	$>$	$<$	$D_{1,<} = D_{2,>}$	$D_{1,>} \subset \mathbf{A}(D_{2,<}), D_{2,<} \subset \mathbf{A}(D_{1,<})$
15	$<$	$<$	$<$	$>$	$D_{1,>} = D_{2,<}$	$D_{1,<} \subset \mathbf{A}(D_{1,>}), D_{2,>} \subset \mathbf{A}(D_{1,<})$
16	$<$	$<$	$<$	$<$	$D_{1,<} = D_{2,<}$	$D_{1,>} \subset \mathbf{A}(D_{1,<}), D_{2,>} \subset \mathbf{A}(D_{1,<})$

Figure 4.7: The control sets and their interrelation to each other. We wrote $p_{>,1}(\tau_1)$ instead of $p_{>,1}$ for emphasizing, that $p_{>,1}(\tau_1) \in \mathcal{X}_\varepsilon^{loc}(\tau_1)$ (and so for the other points).

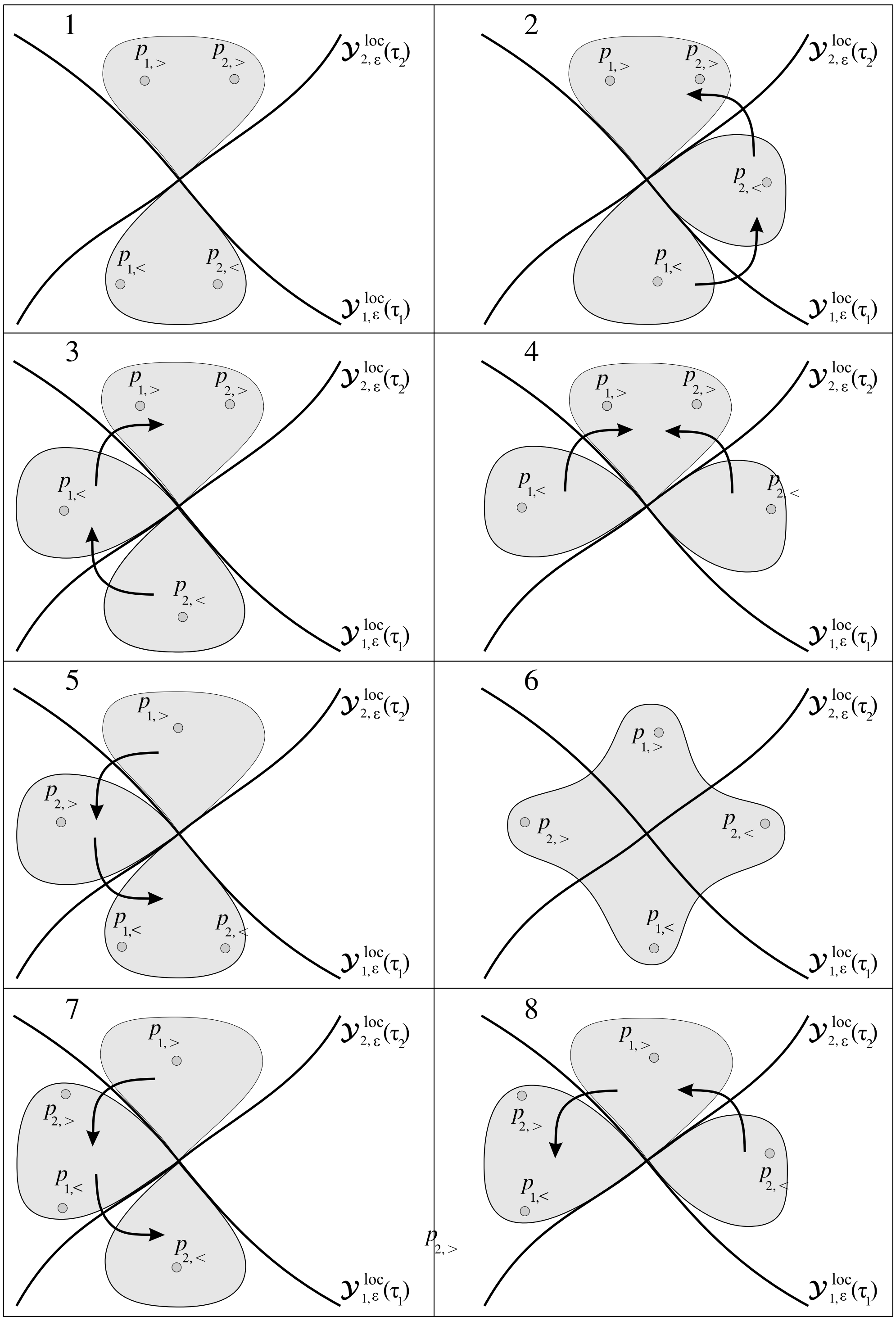

Figure 4.8: Sketch of the control sets and their interrelation, part 1.

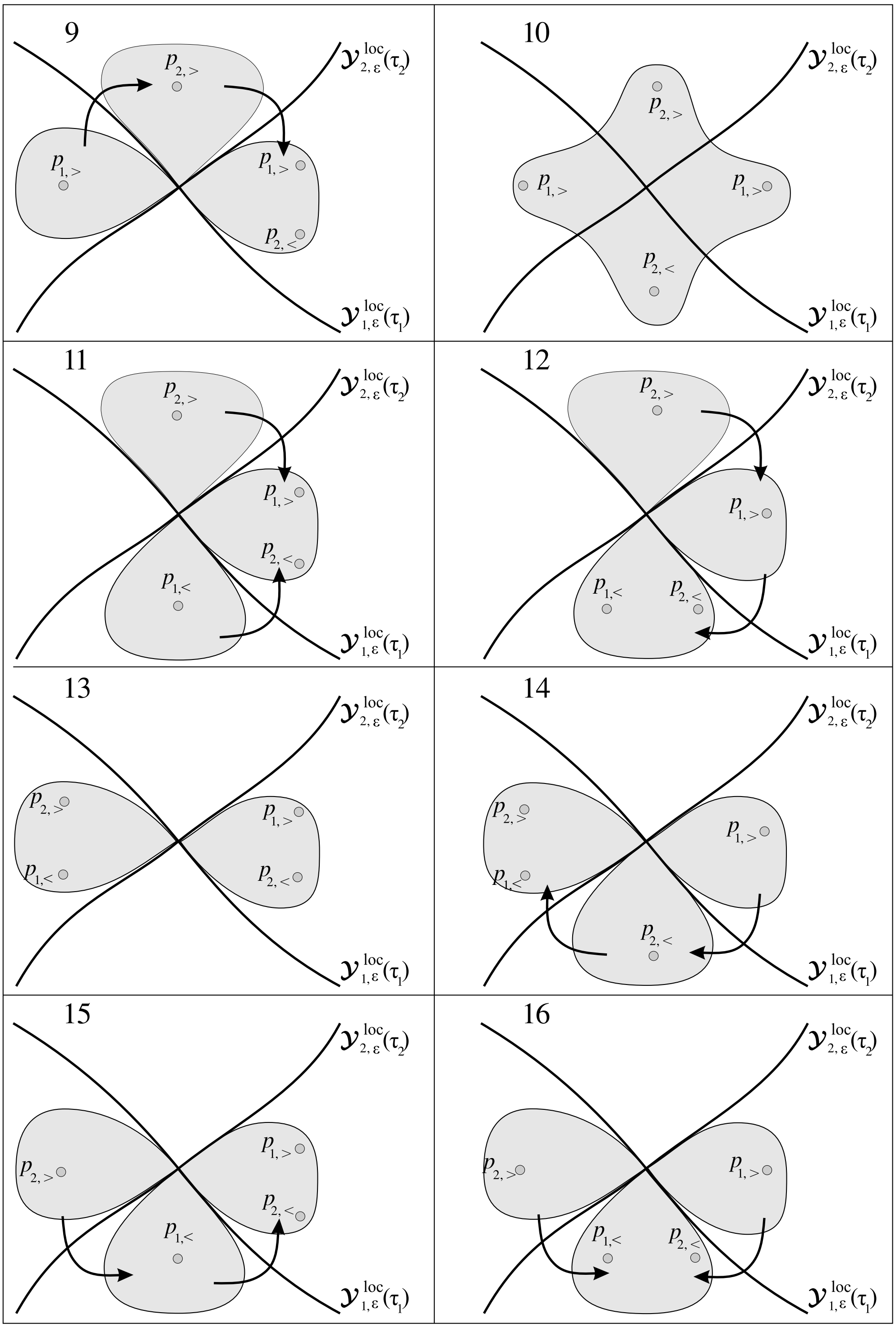

Figure 4.9: Sketch of control sets and their interrelation, part 2.

Proof. Choose $\varepsilon \in (0, \hat{\varepsilon}]$. Then by Theorem 4.1.12 there exists a neighborhood $N \subset \mathbb{R}^d$ of 0 and control sets $D_{j,>}, D_{j,<}, j = 1, 2$ which fulfill (4.51), (4.52) and (4.53). By choosing N eventually smaller we can achieve that the assumptions of Proposition 4.2.4 are fulfilled.

For illustration we prove now the second line of the table, the other lines can be prooved in the same way. Thus we assume, that there are $\tau_1, \tau_2 \in \mathbb{R}, p_{i,>}, p_{i,>} \in \mathcal{X}^{loc}_{i,\varepsilon}(\tau_i), i = 1, 2$ such that (cf. Figure 4.8 (2))

$$p_{1,>} \in \mathcal{X}^{loc}_{1,\varepsilon,>}(\tau_1) \text{ with } p_{1,>} \in \mathcal{X}^{loc}_{2,\varepsilon,>}(\tau_2) \cap N \setminus \{0\}, \tag{4.54}$$

$$p_{2,>} \in \mathcal{X}^{loc}_{2,\varepsilon,>}(\tau_2) \text{ with } p_{2,>} \in \mathcal{X}^{loc}_{1,\varepsilon,>}(\tau_1) \cap N \setminus \{0\}, \tag{4.55}$$

$$p_{1,<} \in \mathcal{X}^{loc}_{1,\varepsilon,<}(\tau_1) \text{ with } p_{1,<} \in \mathcal{X}^{loc}_{2,\varepsilon,<}(\tau_2) \cap N \setminus \{0\}, \tag{4.56}$$

$$p_{2,<} \in \mathcal{X}^{loc}_{2,\varepsilon,>}(\tau_2) \text{ with } p_{2,<} \in \mathcal{X}^{loc}_{1,\varepsilon,>}(\tau_1) \cap N \setminus \{0\}. \tag{4.57}$$

We have to show, that $D_{1,>} = D_{2,>}$. For that purpose we first show, that for every $p \in D_{1,>}$ the relation $D_{2,>} \subset \operatorname{cl} \mathcal{O}^+(p)$ is fulfilled. By (4.51) we have

$$p_{1,>} \in \operatorname{int} D_{1,>},$$
$$p_{2,>} \in \operatorname{int} D_{2,>}.$$

Thus there is a $\sigma > 0$ such that

$$B_\sigma(p_{2,>}) \in \operatorname{int} D_{2,>}.$$

Since by relation (4.54) we have $p_{1,>} \in \mathcal{X}^{loc}_{2,\varepsilon,>}(\tau_2)$ it follows with Proposition 4.2.4 (a), that there is a $T_1 > 0$ and a control function $u_1 \in \mathcal{U}$ with

$$\varphi(T_1, 0, p_{1,>}, u_1) \in B_\sigma(p_{2,>}).$$

By continuous dependency on initial conditions there is an open neighborhood $N_1 \subset \mathbb{R}^d$ of $p_{1,>}$ such that for every $q \in N$

$$\varphi(T_1, 0, q, u_1) \in B_\sigma(p_{2,>}).$$

For every $p \in D_{1,>}$ there is a control function $v \in \mathcal{U}$ and a time $S > 0$ such that

$$\varphi(S, 0, p, v) \in N.$$

By defining

$$w(t) := \begin{cases} 0 & \text{for } t < 0, \\ v(t) & \text{for } t \in [0, S), \\ u_1(t - S) & \text{for } t \in [S, \infty), \end{cases}$$

we get

$$\varphi(T_1 + S, 0, p, w) \in \operatorname{int} D_{2,>}.$$

Thus it follows that

$$D_{2,>} \subset \operatorname{cl} \mathcal{O}^+(p). \tag{4.58}$$

Next we show, that for every $p \in D_{2,>}$ the relation $D_{1,>} \subset \operatorname{cl} \mathcal{O}^+(p)$ is fulfilled. Because $p_{1,>} \in \operatorname{int} D_{1,>}$ there is a $\sigma > 0$ such that $B_\sigma(p_{1,>}) \subset \operatorname{int} D_{1,>}$. Since by relation (4.55) we have $p_{2,>} \in \mathcal{X}^{loc}_{1,\varepsilon,>}(\tau_1)$ it follows with Proposition 4.2.4 (e), that there is a $T_2 > 0$ and a control function $u_2 \in \mathcal{U}$ with

$$\varphi(T_2, 0, p_{2,>}, u_2) \in B_\sigma(p_{1,>}).$$

Because of continuous dependency on initial conditions there is an open neighborhood $N_2 \subset \mathbb{R}^d$ of $p_{2,>}$ such that for every $q \in N_2$

$$\varphi(T_2, 0, q, u_2) \in B_\sigma(p_{1,>}).$$

For every $p \in D_{2,>}$ there is a control function $v \in \mathcal{U}$ and a time $S > 0$ such that

$$\varphi(S, 0, p, v) \in N_2.$$

By defining

$$w(t) := \begin{cases} 0 & \text{for } t < 0, \\ v(t) & \text{for } t \in [0, S), \\ u_2(t - S) & \text{for } t \in [S, \infty), \end{cases}$$

we have

$$\varphi(T_2 + S, 0, p, w) \in \operatorname{int} D_{1,>}$$

Thus it follows that

$$D_{2,>} \subset \operatorname{cl} \mathcal{O}^+(p). \tag{4.59}$$

By (4.59) and (4.58) it follows, that $D_{1,>} = D_{2,>}$.

Next we show, that $D_{1,<} \subset \mathbf{A}(D_{2,<})$. First we choose $\sigma > 0$ such that $B_\sigma(p_{2,<}) \subset \operatorname{int} D_{2,<}(\tau_2)$. Since by relation (4.56) we have $p_{1,<} \subset \mathcal{X}^{loc}_{2,\varepsilon,<}(\tau_2)$, it follows by Proposition 4.2.4 (d), that there is a control function $u_3 \in \mathcal{U}$ and a $T_3 > 0$ such that

$$\varphi(T_3, 0, p_{1,<}, u_3) \in B_\sigma(p_{2,<}) \subset \operatorname{int} D_{2,<}(\tau_2).$$

This means that $p_{1,<} \in \mathbf{A}(D_{2,<})$. Because $p_{1,<}(\tau_1) \in \operatorname{int} D_{1,<}$ for every $p \in D_{1,<}$ there is by continuous dependency on initial values a control function $w \in \mathcal{U}$ and a time $S > 0$ such that $\varphi(S, 0, p, w) \in B_\sigma(p_{2,<}(\tau_2))$. Thus it follows, that $D_{1,<} \subset \mathbf{A}(D_{2,<})$.

Finally we show, that $D_{2,<} \subset \mathbf{A}(D_{1,>})$. First we choose $\sigma > 0$ such that $B_\sigma(p_{1,>}) \subset \operatorname{int} D_{1,>}(\tau_1)$. Since by relation (4.57) we have $p_{2,<} \subset \mathcal{X}^{loc}_{1,\varepsilon,>}(\tau_1)$, it follows by Proposition 4.2.4 (g), that there is a control function $u_4 \in \mathcal{U}$ and a $T_4 > 0$ such that

$$\varphi(T_4, 0, p_{2,<}, u_4) \in B_\sigma(p_{1,>}) \subset \operatorname{int} D_{1,>}(\tau_1).$$

This means that $p_{2,<} \in \mathbf{A}(D_{1,>})$. Because $p_{2,<} \in \operatorname{int} D_{2,<}$ for every $p \in D_{2,<}$ there is by continuous dependency on initial values a control function $w \in \mathcal{U}$ and a time $S > 0$ such that $\varphi(S, 0, p, w) \in B_\sigma(p_{1,>})$. Thus it follows, that $D_{2,<} \subset \mathbf{A}(D_{1,>})$. ■

In Figure 4.8 and 4.9 we sketched the different cases (1)-(16) of Theorem 4.2.5. Here the half-plane above $\mathcal{Y}_{i,\varepsilon}^{loc}(\tau_i)$ is $\mathcal{X}_{i,\varepsilon,>}^{loc}(\tau_i)$ and that under $\mathcal{Y}_i^{loc}(\tau_i)$ is $\mathcal{X}_{i,\varepsilon,<}^{loc}(\tau_i)$ for $i = 1, 2$. The control set $D_{1,>}$ is the one, in which the point $p_{1,>}(\tau_1)$ lies and so on. The arrows show, which of the control sets are in the domain of attraction of another control set. If an arrow starts for example in $D_{1,>}$ and points to $D_{2,>}$ then this means, that $D_{1,>} \subset \mathbf{A}(D_{2,>})$, i.e. $D_{1,>}$ lies in the domain of attraction of $D_{2,>}$.

We have to mention here again, that the Theorem 4.2.5 does *not* show, that two control sets do *not* coincide. For example in the case (2) it could happen, that $D_{1,>} = D_{1,<}$ or $D_{1,>} = D_{2,<}$. This could happen because of global effects 'far away' from the singular point 0, where the behavior of the linearized system has no influence.

4.3 Control Sets and Linearization

In this section we will again consider the nonlinear control system on $\mathbb{R}^d$

$$\begin{array}{l} \dot{x} = f_0(x) + \sum_{i=1}^m u_i(t) f_i(x) \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\} \end{array} \tag{4.60}$$

where U is a compact and convex subset of $\mathbb{R}^m$ and $f_0, \dots, f_m$ are C^2 vector fields on $\mathbb{R}^d$. Suppose that for all $(u, x) \in \mathcal{U} \times \mathbb{R}^d$ the equation (4.60) has an unique solution $\varphi(t, \tau, x, u), t, \tau \in \mathbb{R}$, with $\varphi(\tau, \tau, x, u) = x$.

We suppose, that the system (4.60) has the singular point $x^* = 0 \in \mathbb{R}^d$.

The corresponding linearized system associated with the nonlinear system (4.60) is the bilinear control system on $\mathbb{R}^d$:

$$\begin{array}{l} \dot{x} = A_0 x + \sum_{i=1}^m u_i(t) A_i x \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\} \end{array} \tag{4.61}$$

where $A_i := \left.\frac{\partial f_i}{\partial x}\right|_{x=0}$. We denote the fundamental solution of (4.61) for $u \in \mathcal{U}$ by $\eta(t, \tau, u)$, with $\eta(\tau, \tau, u)x = x$, where $\tau, t \in \mathbb{R}, x \in \mathbb{R}^d$.

Up to now, we only assumed, that there are at least two periodic control functions u^h and $u^s \in \mathcal{U}$ such that the corresponding Lyapunov exponents fulfill Condition 4.1.1. Then we concluded (under additional conditions) in Theorem 4.2.5, that there are two control sets $D_<$ and $D_>$ with nonvoid interior and $0 \in \operatorname{cl} D_> \cap \operatorname{cl} D_<$. Up to now we did not take into account the controllability properties of the bilinear system (4.61).

A necessary condition for controllability of the bilinear system is, that the Lie algebra rank condition

$$\dim \mathcal{LA}\{A_0 + \sum_{i=1}^{m} u_i A_i : u \in U\}(x) = d \tag{4.62}$$

is satisfied for all $x \in \mathbb{R}^d \setminus \{0\}$. This condition implies, that the bilinear system is locally accessible, and that the projected system

$$\begin{array}{l} \dot{p} = h(p, u(t)) = h_0(p) + \sum_{i=1}^{m} u_i(t) h_i(p) \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{locally integrable}\} \end{array} \tag{4.63}$$

with

$$h_i(p) := (A_i - p^T A_i p \cdot \mathrm{id})p \text{ for all } i = 0, \dots, m$$

satisfies the Lie algebra rank condition

$$\dim \mathcal{LA}\{h(\cdot, u) : u \in U\}(p) = d - 1 \text{ for all } p \in \mathbb{P}^{d-1}. \tag{4.64}$$

Thus if (4.62) is fulfilled, both the bilinear and the projected system are locally accessible.

We obtain the following characterization of the controllability properties for the bilinear control system (4.61).

Corollary 4.3.1 *Assume, that the bilinear control system (4.61) fulfills the Lie algebra rank condition (4.62). Let $D \subset \mathbb{P}^{d-1}$ be a main control set of the projected system (4.63).*

1. *If $0 \in \operatorname{int} \Sigma_{Fl}(D)$, then the bilinear system is controllable in the cone $\mathbf{K} := \{\alpha p : p \in \operatorname{int} D, \alpha > 0\}$.*
2. *If $0 \notin \operatorname{int} \Sigma_{Ly}(D)$, then the bilinear system is not controllable in the cone generated by D.*
3. *The bilinear system is controllable in $\mathbb{R}^d \setminus \{0\}$, if and only if the projected system is controllable in $\mathbb{P}^{d-1}$ and $0 \in \operatorname{int} \Sigma_{Ly}(\mathbb{P}^{d-1})$.*

Proof. Cf. Corollary 12.2.6 in [9]. ■

In this section we will show, that the cone $\mathbf{K}$ is a (local) approximation of the control set $D_<$ (and $D_>$) of the nonlinear control system (4.60), if the Lyapunov spectrum of the nonlinear system fulfills an additional condition.

4.3.1 Preliminaries

First we state the following technical lemma.

Lemma 4.3.2 *Let* $0 < \omega_1 < \omega_2$ *and* $(x, y) \in X \times Y$ *with* $x \neq 0$ *and*

$$\|y\| < \omega_1 \|x\|$$

and let $0 < \sigma < \frac{\omega_2 - \omega_1}{1+2\omega_1+\omega_2} \|x\|$. *Then for all* $(x', y') \in B_\sigma(x, y)$ *we have*

$$\|y'\| < \omega_2 \|x'\|.$$

Proof. Let $(x', y') \in B_\sigma(x, y)$ Note that from $\frac{\omega_2-\omega_1}{1+2\omega_1+\omega_2} < 1$ it follows $\|x\| > \sigma$ and therefore $x' \neq 0$. From

$$\begin{aligned}
\frac{\omega_1 \|x'\| + (1+\omega_1)\sigma}{\|x'\|} &\leq \frac{\omega_1 \|x\| + (1+2\omega_1)\sigma}{\|x\| - \sigma} \\
&\leq \frac{\omega_1 \|x\| + (1+2\omega_1)\frac{\omega_2-\omega_1}{1+2\omega_1+\omega_2} \|x\|}{\|x\| - \frac{\omega_2-\omega_1}{1+2\omega_1+\omega_2} \|x\|} \\
&= \frac{\omega_2 + 3\omega_1\omega_2}{1+3\omega_1} \\
&= \omega_2
\end{aligned}$$

we obtain

$$\begin{aligned}
\|y'\| &\leq \|y\| + \sigma \leq \omega_1 \|x\| + \sigma \\
&\leq \omega_1 \|x'\| + (1+\omega_1)\sigma \\
&\leq \omega_2 \|x'\|.
\end{aligned}$$

■

Let $u^s \in \mathcal{U}$ be a periodic control function. Corresponding to u^s we get the nonlinear differential equation

$$\dot{x} = f_0(x) + \sum_{\iota=1}^{m} u_i^s(t) f_i(x) \tag{4.65}$$

which we call the original system. We suppose, that the Lyapunov exponents of the linearized system fulfill

$$0 > \lambda_1^s > \lambda_2^s \geq \ldots \geq \lambda_d^s.$$

We sketch the reduction process of Section 6.2. The goal is to show that in a neighborhood of the origin the trajectories approach the local unstable fibre bundle $\mathcal{X}_\varepsilon^{loc}$ tangentially in a Lipschitz way.

As in Section 6.2 we get by the Floquet transformation $\mathcal{F}$ a transformed system on $X \times Y$ with $\dim X = 1$:

$$\begin{aligned}
\dot{x} &= A^+ x + F^+(t, x, y) \\
\dot{y} &= A^- y + F^-(t, x, y)
\end{aligned}$$

and define

$$X(t) := \mathcal{F}^{-1}(t)X$$
$$Y(t) := \mathcal{F}^{-1}(t)Y.$$

By a radial retraction we get the reduced system,

$$\begin{aligned} \dot{x} &= Ax + F_\varepsilon^+(t,x,y) \\ \dot{y} &= By + F_\varepsilon^-(t,x,y) \end{aligned} \tag{4.66}$$

where we denote the solutions by $\mu_\varepsilon(t,\tau,x,y,u^s)$. Then an $\varepsilon^* > 0$ can be chosen as in Section 6.2 such that for every $\varepsilon \in (0,\varepsilon^*]$ there exists an unstable and a stable fibre bundle

$$\mathcal{X}_\varepsilon = \{(\tau,x,y) \in \mathbb{R}\times X \times Y : y = w_\varepsilon^+(\tau,x)\}$$
$$\mathcal{Y}_\varepsilon = \{(\tau,x,y) \in \mathbb{R}\times X \times Y : x = w_\varepsilon^-(\tau,y)\}$$

for the reduced system (4.66) and local unstable and stable fibre bundles $\mathcal{X}_\varepsilon^{loc}$ and $\mathcal{Y}_\varepsilon^{loc}$ for the original system (4.65). Note that because $\lambda_1 < 0$ the corresponding nonlinear system is locally asymptotically stable in 0. Thus also the trajectories which lie on the unstable manifold tend to zero as time goes to infinity.

We will show now, that the trajectories of the original system (4.65) approach the unstable fibre bundle $\mathcal{X}_\varepsilon^{loc}$ in a Lipschitz way. We prove this assertion in two steps. First we show that the trajectories of the restricted system approach the unstable fibre bundle $\mathcal{X}_\varepsilon$ in a Lipschitz way and then we will show the same for the original system for $\mathcal{X}_\varepsilon^{loc}$.

Lemma 4.3.3 *For every $\omega > 0$ there exists an $\varepsilon(\omega) \in (0,\varepsilon^*]$ such that for all $\varepsilon \in (0,\varepsilon(\omega)]$ and for $\tau \in \mathbb{R}, (x,y) \in X \times Y \setminus \mathcal{Y}_\varepsilon(\tau)$ there is $T > \tau$ with*

$$\mu_\varepsilon(t,\tau,x,y,u^s) \in \{(x,y) \in X \times Y : \|y\| \leq \omega \|x\|\} \text{ for all } t > T.$$

Proof. First choose an $\omega' \in (0,\omega)$. Then there is an $\varepsilon(\omega') > 0$ such that for every $\varepsilon \in (0,\varepsilon(\omega'))$ we have

$$\mathcal{X}_\varepsilon(t) \subset \{(x,y) \in X \times Y : \|y\| \leq \omega' \|x\|\}$$

(cf. the proof of Lemma 6.2.14). For $(x,y) \in X \times Y, \tau \in \mathbb{R}$ we denote the solution of the restricted system (6.34) by

$$\mu_\varepsilon(t,\tau,x,y,u^s) = (\mu_{\varepsilon,X}(t,\tau,x,y,u^s), \mu_{\varepsilon,Y}(t,\tau,x,y,u^s)) \in X \times Y.$$

We have to show, that for $(x,y) \in \mathcal{Y}_\varepsilon(\tau)$ there is a $T > 0$ with

$$\|\mu_{\varepsilon,Y}(t,x,y,u^s)\| < \omega \|\mu_{\varepsilon,X}(t,x,y,u^s)\|.$$

For the definition of $L(\varepsilon)$ cf. Remark 6.2.5. For abbreviation we denote the corresponding Lipschitz constant $L(\varepsilon)$ of F_ε^+ and F_ε^- in the restricted system (4.66) by L. Denote by $\hat{\mathcal{P}}_\varepsilon$ the asymptotic phase for $\mathcal{X}_\varepsilon$. According to Corollary 6.1.21 for every $\gamma \in (\beta + KL, \alpha - KL)$ and every $\tau \in \mathbb{R}, (x,y) \in X \times Y$ we have

$$\|\mu_\varepsilon(t,\tau,x,y,u^s) - \mu_\varepsilon(t,\tau,\mathcal{P}_\varepsilon(\tau,x,y),u^s)\| \leq Qe^{\gamma(t-\tau)} \text{ for } t \geq \tau, \tag{4.67}$$

if we define

$$Q := Q(\tau,x,y) := \frac{K(\gamma-\beta)}{\gamma-\beta-KL} \|(x,y) - \mathcal{P}_\varepsilon(\tau,x,y)\| .$$

Now we choose $\sigma \in (0, \frac{\omega-\omega'}{1+2\omega'+\omega})$ and $\gamma \in (\beta + KL, \alpha - KL)$. Since

$$\mu_\varepsilon(t,\tau,\mathcal{P}_\varepsilon(\tau,x,y),u^s) \in \{(x,y) \in X \times Y : \|y\| \leq \omega' \|x\|\} \text{ for all } t \in \mathbb{R},$$

it suffices according to Lemma 4.3.2, to show that

$$\|\mu_\varepsilon(t,\tau,x,y,u^s) - \mu_\varepsilon(t,\tau,\mathcal{P}_\varepsilon(\tau,x,y),u^s)\| < \sigma \|\mu_{\varepsilon,X}(t,\tau,\mathcal{P}_\varepsilon(\tau,x,y),u^s)\| .$$

Because of (4.67) we have to find a $T > \tau$ with

$$Qe^{\gamma(t-\tau)} < \sigma \|\mu_{\varepsilon,X}(t,\tau,\mathcal{P}_\varepsilon(\tau,x,y),u^s)\| \text{ for all } t > T . \tag{4.68}$$

First show a relation between the trajectory $\mu_\varepsilon(t,\tau,\mathcal{P}(\tau,x,y),u^s)$ which lies on the unstable bundle $\mathcal{X}_\varepsilon(t)$ and its component $\mu_{\varepsilon,X}(t,\tau,\mathcal{P}(\tau,x,y),u^s) \in X$. Because of the Lipschitz continuity of w_ε^+ we get by Theorem 6.1.13

$$\begin{aligned}
\|\mu_\varepsilon(t,\tau,\mathcal{P}_\varepsilon(\tau,x,y),u^s)\| &\leq \|\mu_{\varepsilon,X}(t,\tau,\mathcal{P}_\varepsilon(\tau,x,y),u^s)\| + \|\mu_{\varepsilon,Y}(t,\tau,\mathcal{P}_\varepsilon(\tau,x,y),u^s)\| \\
&= \|\mu_{\varepsilon,X}(t,\tau,\mathcal{P}_\varepsilon(\tau,x,y),u^s)\| + \left\|w_\varepsilon^+(\mu_{\varepsilon,X}(t,\tau,\mathcal{P}_\varepsilon(\tau,x,y),u^s))\right\| \\
&\leq \|\mu_{\varepsilon,X}(t,\tau,\mathcal{P}_\varepsilon(\tau,x,y),u^s)\| + \frac{K^2L}{\delta-KL} \|\mu_{\varepsilon,X}(t,\tau,\mathcal{P}_\varepsilon(\tau,x,y),u^s)\| \\
&= \left(1 + \frac{K^2L}{\delta-KL}\right) \|\mu_{\varepsilon,X}(t,\tau,\mathcal{P}_\varepsilon(\tau,x,y),u^s)\| .
\end{aligned} \tag{4.69}$$

Now by Corollary 6.1.19 for all $\gamma \in (\beta + KL, \alpha - KL)$ we have

$$\|\mu_\varepsilon(t,\tau,\mathcal{P}_\varepsilon(\tau,x,y),u^s)\| \geq \frac{\alpha-\gamma+KL}{K(\alpha-\gamma)} \|\mathcal{P}_\varepsilon(\tau,x,y)\| e^{\gamma(t-\tau)} \text{ for all } t \geq \tau. \tag{4.70}$$

and with (4.69) and (4.70) we obtain

$$\begin{aligned}
\sigma \|\mu_{\varepsilon,X}(t,\tau,\mathcal{P}(\tau,x,y),u^s)\|^s &\geq \sigma \left(1 + \frac{K^2L}{\delta-KL}\right)^{-1} \|\mu_\varepsilon(t,\tau,\mathcal{P}_\varepsilon(\tau,x,y),u^s\| \\
&\geq \sigma \left(1 + \frac{K^2L}{\delta-KL}\right)^{-1} \frac{\alpha-\gamma+KL}{K(\alpha-\gamma)} \|\mathcal{P}_\varepsilon(\tau,x,y))\| e^{\gamma(t-\tau)}.
\end{aligned}$$

For $\gamma' \in (\beta + KL, \alpha - KL)$ with $\gamma' > \gamma$, there is a $T > \tau$ with

$$\sigma \left(1 + \frac{K^2 L}{\delta - KL}\right)^{-1} \frac{\gamma - \beta - KL}{K(\gamma - \beta)} \|\mathcal{P}_\varepsilon(\tau, x, y)\| \, e^{\gamma'(t-\tau)} > Q e^{\gamma(t-\tau)}$$

for all $t > T$ and it follows

$$\sigma \left\|\mu_\varepsilon(t, \tau, \mathcal{P}_\varepsilon(\tau, x, y), u^s)\right\| > Q e^{\gamma(t-\tau)} \text{ for all } t > T.$$

■

After this preparation we can lift the result to the original system. Recall that the cone with angle ω around $X(t)$ is defined by

$$K_\omega(X(t)) = \left\{ v + w \in \mathbb{R}^d : \begin{array}{c} v \in X(t), w \in \mathbb{R}^d \text{ with } \langle v', w\rangle = 0 \text{ for all } v' \in X(t) \\ \text{and } \|w\| \le \omega \|v\| \end{array} \right\}$$

Proposition 4.3.4 *For every $\omega > 0$ there exists an $\tilde{\varepsilon}(\omega) \in (0, \varepsilon^*]$ such that for every $\varepsilon \in (0, \tilde{\varepsilon}(\omega)]$ there exists a neighborhood $W \subset \mathbb{R}^d$ of 0 such that for all $\tau \in \mathbb{R}, p \in W \setminus \mathcal{Y}_\varepsilon^{loc}(\tau)$ there is $T > \tau$ such that*

$$\varphi(t, \tau, p, u^s) \in K_\omega(X(t)) \textit{ for all } t > T.$$

Proof. According to Lemma 6.2.10, for every $\varepsilon \in (0, \varepsilon^*]$ there is a neighborhood $W(\varepsilon) \subset \mathbb{R}^d$ of 0 such that for all $p \in W(\varepsilon)$ we have

$$\varphi(t, \tau, p, u^s) = \mathcal{F}^{-1}(t)\psi(t, \tau, \mathcal{F}(t)p, u^s) = \mathcal{F}^{-1}(t)\mu_\varepsilon(t, \tau, \mathcal{F}(t)p, u^s) \text{ for all } t \ge \tau.$$

According to Lemma 6.2.13, for $\omega > 0$ there exists a $\kappa > 0$ such that for all $t \in \mathbb{R}$

$$\begin{aligned} \mathcal{F}^{-1}(t) \left\{(x, y) \in X \times Y : \ \|y\| \le \kappa \|x\|\right\} &\subset K_\omega(X(t)) \\ \mathcal{F}^{-1}(t) \left\{(x, y) \in X \times Y : \ \|x\| \le \kappa \|y\|\right\} &\subset K_\omega(Y(t)). \end{aligned}$$

From Lemma 4.3.3 we obtain an $\varepsilon(\kappa) > 0$ such that for every $\varepsilon \in (0, \varepsilon(\kappa)]$ and every $\tau \in \mathbb{R}, (x, y) \in X \times Y \setminus \mathcal{Y}_\varepsilon(\tau)$ there is a $T > \tau$, such that

$$\mu_\varepsilon(t, \tau, x, y, u^s) \in \{(x, y) \in X \times Y : \|y\| \le \kappa \|x\|\} \text{ for all } t > T.$$

We define $\tilde{\varepsilon}(\omega) := \varepsilon(\kappa)$ and denote $W := W(\varepsilon)$ for $\varepsilon \in (0, \tilde{\varepsilon}(\kappa)]$. Then for every $\tau \in \mathbb{R}$, $p \in W \setminus \mathcal{Y}_\varepsilon^{loc}(\tau)$ there is a $T > \tau$ such that

$$\begin{aligned} \varphi(t, \tau, p, u^s) &= \mathcal{F}^{-1}(t)\mu_\varepsilon(t, \tau, \mathcal{F}(t)p, u^s) \\ &\in \mathcal{F}^{-1}(t) \left\{(x, y) \in X \times Y : \ \|y\| \le \kappa \|x\|\right\} \subset K_\omega(X(t)) \end{aligned}$$

for every $t > T$. ■

4.3.2 The Linearization Theorem

In the following we characterize control sets of the nonlinear system (4.60) via the eigenspaces of the linearized system. For that purpose we assume, that $\mathcal{U} \times \mathbb{R}^d = \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_l$ with $\mathcal{V}_i$ given by Theorem 1.2.7 such that

$$\begin{aligned} \dim \mathcal{V}_1 = 1 \text{ and } 0 \in \operatorname{int} \Sigma_{Ly}(\mathcal{V}_1), \\ \Sigma_{Ly}(\mathcal{V}_i) \subset \mathbb{R}^- \text{ for } i = 2, \ldots, l. \end{aligned}$$

and

$$\Sigma_{Ly}(\mathcal{V}_1) \cap \Sigma_{Ly}(\mathcal{V}_i) = \emptyset \text{ for } i = 2, \ldots, l. \tag{4.71}$$

The assumption (4.71) guarentees, that then every periodic control function (such that the corresponding Lyapunov exponents are not 0), has a local unstable fibre bundle.

Notation:
For $g \in \mathcal{S}$ we denote by $u_g \in \mathcal{U}$ the corresponding periodic and piecewise constant control function with period Θ_g (cf. Section 1.2.1). Then we can apply the reduction process from Section 6.2 on the differential equation

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_{g,i}(t) f_i(x).$$

We will now supply the resulting objects (for example the stable and unstable fibre bundles) with subscript $_g$. We denote by $\sigma_{g,1}, \ldots, \sigma_{g,d}$ the eigenvalues of an element $g \in \mathcal{S}$ where we enorder the eigenvalues corresponding to their multiplicities. The Lyapunov exponents of the corresponding linear differential equation with control function u_g are $\lambda(u_g) = \{\mathrm{Re}(\frac{1}{\Theta} \ln \sigma_{i,g}), i = 1, \ldots, d\}$ (compare Section 6.2.1). We numerate the Lyapunov exponents, such that $\lambda_{g,i} = \mathrm{Re}(\frac{1}{\Theta} \ln \sigma_{i,g})$ and

$$\lambda_{g,1} > \lambda_{g,2} \geq \ldots \geq \lambda_{g,d}.$$

We suppose from now on, that the Lyapunov exponents $\lambda_{g,i}$ and the eigenvalues $\sigma_{g,i}$ are ordered in this way. We denote by $E(\sigma_{g,i}) = E(\lambda_{g,i})$ the corresponding generalized real eigenspace of g which belongs to $\sigma_{g,i}$. Note that we have

$$X_g(0) = E(\sigma_{g,1}) \text{ and } Y_g(0) = \oplus_{i=2}^{d} E(\sigma_{g,i}),$$

where the sets $X_g(t)$ and $Y_g(t)$ are defined as in (6.27).

Theorem 4.3.5 *Consider the nonlinear control system*

$$\begin{aligned} &\dot{x} = f_0(x) + \textstyle\sum_{i=1}^{m} u_i(t) f_i(x) \\ &u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{ locally integrable}\} \end{aligned} \tag{4.72}$$

where U is a compact and convex subset of $\mathbb{R}^m$ and $f_0, \ldots, f_m$ are C^2 vector fields on $\mathbb{R}$. Suppose that for all $(u, x) \in \mathcal{U} \times \mathbb{R}^d$ the equation (4.72) has a unique solution $\varphi(t, \tau, x, u), t, \tau \in \mathbb{R}$ with $\varphi(\tau, \tau, x, u) = x$. Assume, that following properties are satisfied.

(1) *The nonlinear control system (4.72) has one singular point* $x^* = 0 \in \mathbb{R}^d$*, and is Lie-determined such that* $\mathbb{R}^d \setminus \{0\}$ *and* $\{0\}$ *are maximal integral manifolds.*

(2) *The projected system (4.63) fulfills the Lie algebra rank condition (4.64).*

(3) *Let* $\mathcal{U} \times \mathbb{R}^d = \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_l$ *defined as in Theorem 1.2.7 such that*

$$\begin{aligned}\dim \mathcal{V}_1 = 1 \textit{ and } 0 \in \operatorname{int} \Sigma_{Ly}(\mathcal{V}_1),\\ \Sigma_{Ly}(\mathcal{V}_i) \subset \mathbb{R}^- \textit{ for } i = 2, \ldots, l.\end{aligned}$$

and

$$\Sigma_{Ly}(\mathcal{V}_1) \cap \Sigma_{Ly}(\mathcal{V}_i) = \emptyset \textit{ for } i = 2, \ldots, l.$$

(4) *There is a neighborhood* $V \subset \mathbb{R}^d$ *of* 0 *such that for every* $x \in V \setminus \{0\}, t \in \mathbb{R}$ *and every piecewise constant periodic* $u \in \mathcal{U}$ *the pair* $(u(t+\cdot), x)$ *is a strong inner pair.*

Then there exist two control sets $D_>, D_< \subset \mathbb{R}^d$ *(which could be identical) with nonvoid interior such that for all* $s > 0$*, all* $g \in \operatorname{int} \mathcal{S}_{\leq s}$ *and all* $\omega > 0$ *there is a* $\hat{\sigma} := \hat{\sigma}(\omega, g)$ *such that if* $\lambda_{g,1} > 0$ *then*

$$\begin{aligned}D_> \cap B_\sigma(0) \cap K_\omega(E(\lambda_{g,1})) \neq \emptyset \textit{ and}\\ D_< \cap B_\sigma(0) \cap K_\omega(E(\lambda_{g,1})) \neq \emptyset \textit{ for all } \sigma \in (0, \hat{\sigma})\end{aligned} \tag{4.73}$$

and if $\lambda_{g,1} < 0$

$$\begin{aligned}D_> \cap B_\sigma(0) \cap K_\omega(E(\lambda_{g,1})) \neq \emptyset \textit{ or}\\ D_< \cap B_\sigma(0) \cap K_\omega(E(\lambda_{g,1})) \neq \emptyset \textit{ for all } \sigma \in (0, \hat{\sigma}).\end{aligned} \tag{4.74}$$

Furthermore, if $\lambda_{g,1} > 0$ *there is an* $\varepsilon > 0$ *such that there are* $p_> \in \mathcal{X}^{loc}_{g,\varepsilon,>}(0) \cap \mathcal{X}^{loc}_{g,\varepsilon}(0)$, $p_< \in \mathcal{X}^{loc}_{g,\varepsilon,<}(0) \cap \mathcal{X}^{loc}_{g,\varepsilon}(0)$ *with*

$$\begin{aligned}\{\varphi(t, 0, p_>, u_g) : t \leq 0\} \subset \operatorname{int} D_>,\\ \{\varphi(t, 0, p_<, u_g) : t \leq 0\} \subset \operatorname{int} D_<.\end{aligned} \tag{4.75}$$

Proof. First choose u_1, u_2 which correspond to $g_1, g_2 \in \operatorname{int} \mathcal{S}_{\leq s}$ for some $s > 0$, such that the Lyapunov exponents fulfill

$$\begin{aligned}\lambda_{1,1} > \quad 0 \quad > \lambda_{1,2} \geq \ldots \lambda_{1,d} \text{ and}\\ \lambda_{2,1} > \quad 0 \quad > \lambda_{2,2} \geq \ldots \lambda_{2,d}.\end{aligned}$$

Then, by Theorem 4.2.5, there exists a neighborhood $W \subset \mathbb{R}^d$ and four control sets $D_{j,>}, D_{j,<} \subset \mathbb{R}^d, j = 1, 2$ with nonvoid interior such that for every $\tau \in \mathbb{R}$ we have

$$\{\varphi(t, 0, p_{>,j}, u_j(\tau + \cdot)) : t < 0\} \subset \operatorname{int} D_{j,>}$$

for every $p_{>,j} \in \mathcal{X}^{loc}_{j,\varepsilon}(\tau)$ with $p_{>,j} \in \mathcal{X}^{loc}_{j,\varepsilon,>}(\tau) \cap W \setminus \{0\}$ and

$$\{\varphi(t, 0, p_{<,j}, u_j(\tau + \cdot)) : t < 0\} \subset \operatorname{int} D_{j,<}$$

for every $p_{<,j} \in \mathcal{X}^{loc}_{j,\varepsilon}(\tau)$ with $p_{<,j} \in \mathcal{X}^{loc}_{j,\varepsilon,<}(\tau) \cap W \setminus \{0\}$ for $j = 1, 2$. In particular, we obtain

$$x^* \in \operatorname{cl} D_{1,>} \cap \operatorname{cl} D_{1,<} \cap \operatorname{cl} D_{2,>} \cap \operatorname{cl} D_{2,<}.$$

Next we show $D_{1,>} = D_{2,>}$ and $D_{1,<} = D_{2,<}$. We denote by $\sigma_{j,1}, \dots, \sigma_{j,d}$ the eigenvalues of $g_j, j = 1, 2$. By Lemma 6.2.14, for every $\omega > 0$ there is an $\varepsilon(\omega) > 0$ such that for $j = 1, 2$

$$\begin{aligned} &\mathcal{X}^{loc}_{j,\varepsilon}(t) \subset K_\omega(X_j(t)) \text{ and} \\ &\mathcal{Y}^{loc}_{j,\varepsilon}(t) \subset K_\omega(Y_j(t)) \end{aligned} \tag{4.76}$$

for all $\varepsilon \in (0, \varepsilon(\omega)]$ and $t \in \mathbb{R}$.

The projected system (defined as in (4.63)) has chain control sets $E_1, \dots, E_l$ which are ordered according to the decomposition $\mathcal{V}_1, \dots, \mathcal{V}_l$, i.e. $\dim E_1 = 1$. Denote by $D_1 \prec \dots \prec D_k =: C$ the main control sets of the projected system with $\operatorname{cl} C = E_1$. Since $g_i \in \operatorname{int} S_{\leq s}$ and $\lambda_{i,1} > \lambda_{i,j}$ for $j = 2, \dots, d, i = 1, 2$ we get

$$\begin{aligned} &\mathbb{P}E(\sigma_{1,1}) \in \operatorname{int} C, \\ &\mathbb{P}E(\sigma_{2,1}) \in \operatorname{int} C \end{aligned}$$

Thus by Proposition 1.2.14 it follows

$$\begin{aligned} E(\sigma_{1,1}) \cap \oplus_{j=2}^d E(\sigma_{2,j}) = \{0\}, \\ E(\sigma_{2,1}) \cap \oplus_{j=2}^d E(\sigma_{1,j}) = \{0\}. \end{aligned} \tag{4.77}$$

If we define $X_i(t)$ and $Y_i(t)$ as in (6.27) then we have

$$\begin{aligned} X_i(0) &= E(\sigma_{i,1}), \\ Y_i(0) &= \oplus_{j=2}^d E(\sigma_{i,j}). \end{aligned}$$

Thus by relation (4.77) we get

$$\begin{aligned} X_1(0) \cap Y_2(0) = 0, \\ X_2(0) \cap Y_1(0) = 0. \end{aligned}$$

Choose $\omega > 0$ such that

$$\begin{aligned} &K_\omega(X_1(0)) \cap K_\omega(Y_2(0)) = \{0\} \text{ and} \\ &K_\omega(X_2(0)) \cap K_\omega(Y_1(0)) = \{0\}. \end{aligned} \tag{4.78}$$

Hence by relation (4.76)

$$\begin{aligned} &\mathcal{X}^{loc}_{1,\varepsilon}(0) \cap \mathcal{Y}^{loc}_{2,\varepsilon}(0) = \{0\} \text{ and} \\ &\mathcal{X}^{loc}_{2,\varepsilon}(0) \cap \mathcal{Y}^{loc}_{1,\varepsilon}(0) = \{0\}, \end{aligned}$$

for $\varepsilon \in (0, \varepsilon(\omega))$. Because of (4.78) we have two possibilities. Either

$$\begin{aligned} \mathcal{X}^{loc}_{1,\varepsilon}(0) \cap \mathcal{X}^{loc}_{1,\varepsilon,>}(0) \subset \mathcal{X}^{loc}_{2,\varepsilon,>}(0), \\ \mathcal{X}^{loc}_{1,\varepsilon}(0) \cap \mathcal{X}^{loc}_{1,\varepsilon,<}(0) \subset \mathcal{X}^{loc}_{2,\varepsilon,<}(0), \end{aligned} \tag{4.79}$$

or

$$\begin{aligned} \mathcal{X}^{loc}_{1,\varepsilon}(0) \cap \mathcal{X}^{loc}_{1,\varepsilon,>}(0) \subset \mathcal{X}^{loc}_{2,\varepsilon,<}(0), \\ \mathcal{X}^{loc}_{1,\varepsilon}(0) \cap \mathcal{X}^{loc}_{1,\varepsilon,<}(0) \subset \mathcal{X}^{loc}_{2,\varepsilon,>}(0). \end{aligned} \tag{4.80}$$

If relation (4.80) holds, then we interchange the notations of $\mathcal{X}^{loc}_{1,\varepsilon,>}$ and $\mathcal{X}^{loc}_{1,\varepsilon,<}$ such that relation (4.79) holds. After making this notational modification we can find

$$\begin{aligned} p_{1,<} \in \mathcal{X}^{loc}_{1,\varepsilon,<}(0) \cap \mathcal{X}^{loc}_{1,\varepsilon}(0) \text{ with } p_{1,<} \in \mathcal{X}^{loc}_{2,\varepsilon,<}(0), \\ p_{1,>} \in \mathcal{X}^{loc}_{1,\varepsilon,>}(0) \cap \mathcal{X}^{loc}_{1,\varepsilon}(0) \text{ with } p_{1,>} \in \mathcal{X}^{loc}_{2,\varepsilon,>}(0). \end{aligned}$$

Now there are two more possibilities: Either

$$\begin{aligned} \mathcal{X}^{loc}_{2,\varepsilon}(0) \cap \mathcal{X}^{loc}_{2,\varepsilon,>}(0) \subset \mathcal{X}^{loc}_{1,\varepsilon,>}(0), \\ \mathcal{X}^{loc}_{2,\varepsilon}(0) \cap \mathcal{X}^{loc}_{2,\varepsilon,<}(0) \subset \mathcal{X}^{loc}_{1,\varepsilon,<}(0), \end{aligned} \tag{4.81}$$

or

$$\begin{aligned} \mathcal{X}^{loc}_{2,\varepsilon}(0) \cap \mathcal{X}^{loc}_{2,\varepsilon,<}(0) \subset \mathcal{X}^{loc}_{1,\varepsilon,>}(0), \\ \mathcal{X}^{loc}_{2,\varepsilon}(0) \cap \mathcal{X}^{loc}_{2,\varepsilon,>}(0) \subset \mathcal{X}^{loc}_{1,\varepsilon,<}(0). \end{aligned} \tag{4.82}$$

If the relations (4.81) hold we can find

$$\begin{aligned} p_{2,>} \in \mathcal{X}^{loc}_{2,\varepsilon,>}(0) \cap \mathcal{X}^{loc}_{2,\varepsilon}(0) \text{ with } p_{2,>} \in \mathcal{X}^{loc}_{1,\varepsilon,>}(0), \\ p_{2,<} \in \mathcal{X}^{loc}_{2,\varepsilon,<}(0) \cap \mathcal{X}^{loc}_{2,\varepsilon}(0) \text{ with } p_{2,<} \in \mathcal{X}^{loc}_{1,\varepsilon,<}(0). \end{aligned}$$

Thus by Theorem 4.2.5 case (1) it follows, that $D_{1,<} = D_{2,<}$ and $D_{1,>} = D_{2,>}$ and the relation (4.75).

If relation (4.82) is valid, then we can find

$$\begin{aligned} p_{2,>} \in \mathcal{X}^{loc}_{2,\varepsilon,>}(0) \cap \mathcal{X}^{loc}_{2,\varepsilon}(0) \text{ with } p_{2,>} \in \mathcal{X}^{loc}_{1,\varepsilon,<}(0), \\ p_{2,<} \in \mathcal{X}^{loc}_{2,\varepsilon,<}(0) \cap \mathcal{X}^{loc}_{2,\varepsilon}(0) \text{ with } p_{2,<} \in \mathcal{X}^{loc}_{1,\varepsilon,>}(0). \end{aligned}$$

Thus by Theorem 4.2.5, case (10), it follows, that $D_{1,<} = D_{2,<}$ and $D_{1,>} = D_{2,>}$ and the relation (4.75).

Now define

$$D_< := D_{1,<} = D_{2,<} \text{ and } D_> := D_{1,>} = D_{2,>}.$$

We have shown that for every $s > 0, g \in \operatorname{int} \mathcal{S}_{\leq s}$ with $\lambda_{g,1} > 0$ there is an $\varepsilon > 0$ and $p_> \in \mathcal{X}^{loc}_{g,\varepsilon,>}(0) \cap \mathcal{X}^{loc}_{g,\varepsilon}(0)$, $p_< \in \mathcal{X}^{loc}_{g,\varepsilon,<}(0) \cap \mathcal{X}^{loc}_{g,\varepsilon}(0)$ with

$$\begin{aligned} \{\varphi(t, 0, p_>, u_g) : t \leq 0\} \subset \operatorname{int} D_>, \\ \{\varphi(t, 0, p_<, u_g) : t \leq 0\} \subset \operatorname{int} D_<. \end{aligned}$$

From relation (4.76) we get that there exists $\varepsilon(\omega) > 0, \omega > 0$ such that

$$\mathcal{X}^{loc}_{g,\varepsilon}(0) \subset K_\omega(X_g(0)) = K_\omega(E(\lambda_{g,1}))$$

for all $\varepsilon \in (0, \varepsilon(\omega)]$ and relation (4.73) follows.

Next we show the relations (4.74). For some $t > 0$ choose $g_1, g_2 \in \operatorname{int} \mathcal{S}_{\leq s}$ and denote by $u_1, u_2 \in \mathcal{U}$ the corresponding piecewise constant and periodic control function with Lyapunov exponents

$$\begin{array}{rll} \lambda_{1,1} > & 0 > & \lambda_{1,2} \geq \ldots \geq \lambda_{1,d} \qquad \text{and} \\ & 0 > & \lambda_{2,1} > \lambda_{2,2} \geq \ldots \geq \lambda_{2,d}. \end{array}$$

We denote the local stable and unstable fibre bundles corresponding to g_j by $\mathcal{X}^{loc}_{j,\varepsilon}$ and $\mathcal{Y}^{loc}_{j,\varepsilon}$ for $\varepsilon \in (0, \hat{\varepsilon}_j], j = 1, 2$ (where $\hat{\varepsilon}_j$ is defined as in Proposition 6.2.11).

By Proposition 4.3.4 for every $\omega > 0$ there exists an $\varepsilon(\omega) \in (0, \min\{\hat{\varepsilon}_1, \hat{\varepsilon}_2\}]$ such that for every $\varepsilon \in (0, \varepsilon(\omega)]$ there exists a neighborhood $W \subset \mathbb{R}^d$ of 0 such that for all $\tau \in \mathbb{R}, p \in W \setminus \mathcal{Y}^{loc}_{2,\varepsilon}(\tau)$ there is a $T > \tau$ such that

$$\varphi(t, \tau, p, u^s) \in K_\omega(X_2(t)) \text{ for all } t > T. \tag{4.83}$$

Because $X_1(0) \cap Y_2(0) = 0$ we can choose $\omega > 0$ small enough, such that

$$\mathcal{X}^{loc}_{1,\varepsilon}(0) \cap \mathcal{Y}^{loc}_{2,\varepsilon}(0) = \{0\}$$

for all $\varepsilon \in (0, \varepsilon(\omega))$. Now by the general Existence Theorem 3.7.1 for all $\varepsilon \in (0, \varepsilon(\omega))$ there is a $p \in \mathcal{X}^{loc}_{1,\varepsilon}(0) \cap W$ such that

$$\varphi(t, 0, p, u_2) \in \operatorname{int} D \text{ for } t \geq 0 \text{ and} \tag{4.84}$$

$$\varphi(t, 0, p, u_1) \in \operatorname{int} D \text{ for } t < 0. \tag{4.85}$$

Because of (4.85) it follows, that $D = D_>$ or $D = D_<$. Thus by (4.83) there is a $T > 0$ such that

$$\varphi(t, 0, p, u_2) \in K_\omega(X(t)) \text{ for all } t > T$$

By periodicity of $X(\cdot)$ it follows that there is a $\hat{\sigma} = \sigma(\omega, g_2)$ such that for every $\sigma \in (0, \hat{\sigma})$ we can find a $t > T$ with

$$\varphi(t, 0, p, u_2) \in B_\sigma(0) \cap K_\omega(X(t)) = B_\sigma(0) \cap K_\omega(E(\lambda_{2,1}))$$

and relation (4.74) follows because $\varphi(t, 0, p, u^s) \in \operatorname{int} D_>$ or $\in \operatorname{int} D_<$ for $t > T$. ■

For illustration of the preceding theorem consider Figure 4.10. For some $g \in \operatorname{int} \mathcal{S}_{\leq s}, s > 0$, with $\lambda_{g,1} > 0$ it shows the corresponding eigenspace $E(\lambda_{g,1})$ of the biggest Lyapunov exponent. Theorem 4.3.5 states that for a given cone $K_\omega(E(\lambda_{g,1}))$ around $E(\lambda_{g,1})$ we can find a ball with radius $\sigma > 0$ such that $D_> \cap B_\sigma(0) \cap K_\omega(E(\lambda_{g,1})) \neq \emptyset$ and $D_< \cap B_\sigma(0) \cap K_\omega(E(\lambda_{g,1})) \neq \emptyset$. Choosing ω sufficiently small, we see that the control

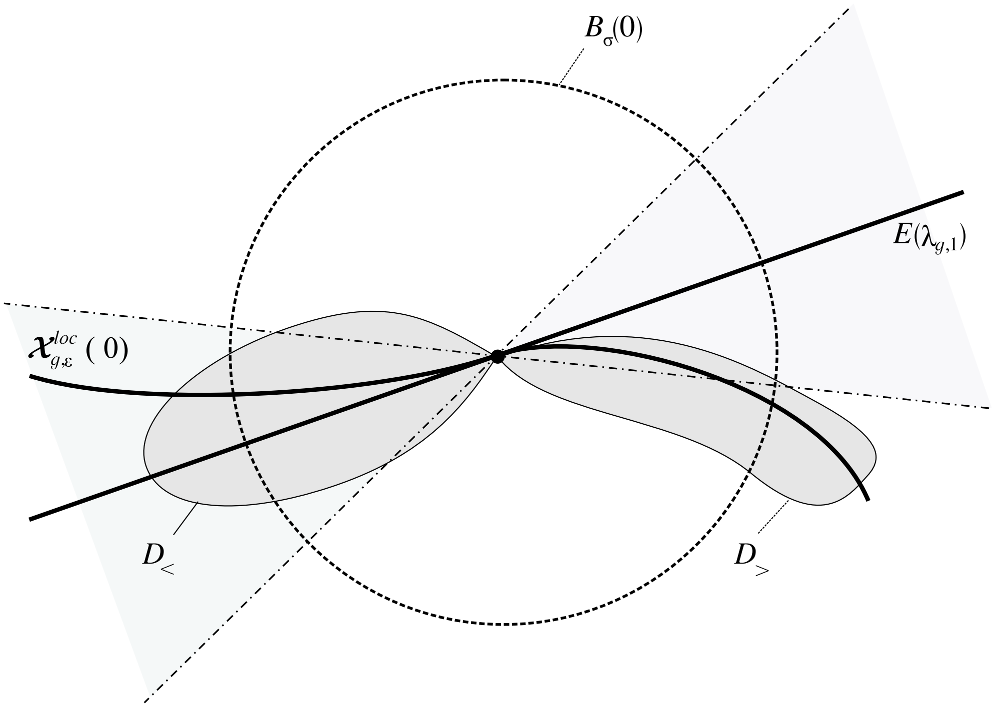

Figure 4.10: The cone $K_\omega(E(\lambda_{g,1}))$ togther with the control set $D_>$ and $D_<$.

sets actually gets arbitrary close to $E(\lambda_{g,1})$. The trivial case, where $E(\lambda_{g,1}) \cap B_\sigma(0) \subset D_<$ is also included.

If we denote by $C \subset \mathbb{P}^{d-1}$ the main control set of the projected system with $0 \in \operatorname{int} \Sigma_{Ly}(C)$, then because $g \in \operatorname{int} \mathcal{S}_{\leq s}$, we have $\mathbb{P}E(\lambda_{g,1}) \in C$. Thus by Corollary 4.3.1 the control sets $D_>$ and $D_<$ are actually 'tangential' in a Lipschitz way to the controllability regions $\mathbf{K}$ of the linearized system.

Chapter 5

Examples

In this chapter we want to apply the results of the previous chapters tp two control systems with a singular point at the origin.

First we consider the Duffing-van der Pol oscillator, which we turn into a control system by perturbation of one parameter of the equation. We observe that for small control range the system does not possess a control set with nonvoid interior near the origin. This can be explained by Theorem 2.1.4, because the Lyapunov spectrum of the system has only negative values.

If the control range exceeds a certain level, then we get positive Lyapunov exponents. By applying Theorem 4.1.12 we conclude that there are two control sets with nonvoid interior such that the origin lies in their closure. This theoretical result is illustrated by numerical computation of the control sets.

We were able to compute the Lyapunov spectrum of the Duffing-van der Pol oscillator analytically, because it is a two-dimensional system. For the second example, the perturbed Lorenz equation, this is not possible, because it is a three-dimensional system. But for showing the existence of control sets we do not need the whole Lyapunov spectrum. For the perturbed Lorenz equation we compute the eigenvalues of the linearized system, corresponding to constant control functions. Then it turns out, that there are constant control functions such that the assumptions of the Existence Theorem 4.1.12 are fulfilled. We conclude that there are control sets with nonvoid interior near the origin and compute the control sets.

5.1 The Duffing-van der Pol oscillator

Consider the deterministic Duffing-van der Pol oscillator

$$\begin{cases} \dot{x} = y \\ \dot{y} = \alpha x - \beta y - x^3 - x^2 y \end{cases}$$

with $\alpha, \beta \in \mathbb{R}$. It exhibits non-degenerate local and global bifurcation behavior. The Duffing-van der Pol oscillator is important for modeling physical phenomena. Since its nonlinear structure is relatively simple, its local and global bifurcation behavior has been studied in depth, cf. for example Guckenheimer and Holmes [15].

If we perturb the parameter α time variant, then this can be interpreted as a control system of the following form

$$\begin{cases} \dot{x} = y \\ \dot{y} = (\alpha + u(t))x - \beta y - x^3 - x^2 y \end{cases} \tag{5.1}$$
$$u \in \mathcal{U}_\rho := \{u : \mathbb{R} \to \mathbb{R} : u(t) \in U_\rho \text{ for a.a. } t \in \mathbb{R}, \text{ locally integrable}\}$$

where $U_\rho = [-\rho, \rho] \subset \mathbb{R}$ is compact, and $\rho \in [0, \rho^{\max}]$ for a $\rho^{\max} > 0$ and with the parameters $\alpha, \beta \in \mathbb{R}$ fixed. The control range U_ρ models the intensity of the allowed perturbations.

The control system (5.1) has the singular point

$$x^* = 0 \in \mathbb{R}^2$$

which is independent of the parameters α and β.

In the following we want to investigate for given $\alpha, \beta \in \mathbb{R}$ if there are control sets near the singular point 0. This will depend on the chosen $\rho \in [0, \rho^{\max}]$ which determines the intensity of the perturbations by U_ρ. We will apply the Theorems 3.7.1 and 4.3.5 to get the existence of control sets $D \subset \mathbb{R}^2$ with nonvoid interior and $x^* \in \operatorname{cl} D$ and will verify the results numerically.

Before we can apply theses Theorems, we have to verify, that the control system (5.1) really fulfills the assumptions of these Theorems.

We first show that the system (5.1) is locally accessible on $\mathbb{R}^2 \setminus \{0\}$ for all $\alpha, \beta \in \mathbb{R}$. If we use the coordinates with respect to the canonical basis $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ in the tangent space of $\mathbb{R}^2$, the system (5.1) can be written in the form

$$\begin{array}{l} \dot{x} = f_0(x) + u(t) f_1(x) \\ u \in \mathcal{U}_\rho := \{u : \mathbb{R} \to \mathbb{R} : u(t) \in U_\rho \text{ for a.a. } t \in \mathbb{R}, \text{ locally integrable}\} \end{array} \tag{5.2}$$

with

$$\begin{array}{l} f_0(x) = x_2 \frac{\partial}{\partial x_1} + (\alpha x_1 - \beta - x_1^3 - x_1^2 x_2) \frac{\partial}{\partial x_2} \\ f_1(x) = x_1 \frac{\partial}{\partial x_2}. \end{array}$$

We have to verify the Lie algebra rank condition

$$\dim \Delta_{\mathcal{LA}}(x) = 2 \text{ for all } x \in \mathbb{R}^2 \setminus \{0\},$$

where

$$\mathcal{LA} := \mathcal{LA}\{f_0 + u_1 f_1 : u \in U\}$$

is the Lie algebra spanned by the vector fields f_0 and f_1. This means, that we have to show that

$$\dim \operatorname{span}\{\operatorname{ad}_{f_i}^k f_j : k \in \mathbb{N}_0 \text{ and } i, j \in \{0, 1\}\}(x) = 2 \text{ for all } x \in \mathbb{R}^2 \setminus \{0\}. \tag{5.3}$$

Remark 5.1.1 *We repeat here the definition of the* ad*-operator for completeness. If we have two smooth vector fields X and Y on a smooth m-dimensional manifold M, then*

$$\mathrm{ad}_X^0 Y = Y \text{ and } \mathrm{ad}_X^k Y = [X, \mathrm{ad}_X^{k-1} Y] \text{ for } k \geq 1.$$

Here $[X,Y]$ denotes the Lie bracket of the two vector fields X and Y. If we have $X(x) = \sum_{i=1}^m \alpha_i(x)\frac{\partial}{\partial x_i}$ and $Y(x) = \beta_i(x)\frac{\partial}{\partial x_i}$ in local coordinates with smooth function $\alpha_i, \beta_i : M \to \mathbb{R}$, then

$$[X,Y](x) = \sum_{j=1}^m \left[\sum_{i=1}^m \left(\alpha_i(x)\frac{\partial \beta_j}{\partial x_i}(x) - \beta_i(x)\frac{\partial \alpha_j}{\partial x_i}(x) \right) \right] \frac{\partial}{\partial x_j}.$$

For notational convenience and for better readability we identify x with $\frac{\partial}{\partial x_1}$ and y with $\frac{\partial}{\partial x_2}$ and write the system (5.2) in another form:

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= f_0 \begin{pmatrix} x \\ y \end{pmatrix} + u(t) f_1 \begin{pmatrix} x \\ y \end{pmatrix} \\ u \in \mathcal{U}_\rho &:= \{u : \mathbb{R} \to \mathbb{R} : u(t) \in U_\rho \text{ for a.a. } t \in \mathbb{R}, \text{ locally integrable}\} \end{aligned} \tag{5.4}$$

with

$$\begin{aligned} f_0 \begin{pmatrix} x \\ y \end{pmatrix} &:= \begin{pmatrix} y \\ \alpha x - \beta y - x^3 - x^2 y \end{pmatrix} \\ f_1 \begin{pmatrix} x \\ y \end{pmatrix} &:= \begin{pmatrix} 0 \\ x \end{pmatrix}. \end{aligned}$$

By computing

$$\mathrm{ad}_{f_0} f_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y - xb + x^3 \end{pmatrix}$$

one easily sees, that

$$\dim \mathrm{span}\{f_0, f_1, \mathrm{ad}_{f_0} f_1\}(x,y)^T = 2 \text{ for all } (x,y)^T \in \mathbb{R}^2, (x,y) \neq (0,0).$$

This means, that the Lie algebra rank (5.3) condition is fulfilled for all $(x,y)^T \in \mathbb{R}^2, (x,y) \neq (0,0)$. Thus the system (5.4) is locally accessible from all points in $\mathbb{R}^2$ except for $x^* = (0,0)^T$.

Next we show, that for all $u \in \mathrm{int}\, U_\rho, \rho > 0$ and all $(x,y) \neq (0,0)$ the pairs $(u, (x,y)^T) \neq 0$ are strong inner pairs. For doing this, we fix $u \in U$ and define

$$f \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} y \\ (\alpha + u)x + \beta y - x^3 - x^2 y \end{pmatrix}.$$

We have to show, that the controllability rank condition

$$\dim \mathrm{span}\{(f, ad_f^k f_1) : k = 0, 1, \dots\}(x,y)^T = 2 \tag{5.5}$$

is fulfilled (cf. Proposition 1.1.18 and Corollary 1.1.19). We compute

$$\mathrm{ad}_f\, f_1(x,y) = \begin{pmatrix} -x \\ y - xb + x^3 \end{pmatrix}.$$

Then we have to look at the following three different cases.

- For $(x,0) \in \mathbb{R}^2, x \neq 0$

$$\dim \operatorname{span}\{f, \mathrm{ad}\, f_1\}(x,0) = \dim \operatorname{span}\left\{ \begin{pmatrix} 0 \\ (\alpha+u)x - x^3 \end{pmatrix} \begin{pmatrix} -x \\ -xb + x^3 \end{pmatrix} \right\} = 2.$$

- For $(0,y) \in \mathbb{R}^2, y \neq 0$

$$\dim \operatorname{span}\{f, \mathrm{ad}_f\, f_1\}(0,y) = \operatorname{span}\left\{ \begin{pmatrix} y \\ \beta y \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} \right\} = 2.$$

- For $(x,y) \in \mathbb{R}^2, (x,y) \neq (0,0)$

$$\dim \operatorname{span}\{f_1, \mathrm{ad}_f\, f_1\}(x,y) = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} \begin{pmatrix} -x \\ y - xb + x^3 \end{pmatrix} \right\} = 2.$$

Thus the controllability rank condition (5.5) is fulfilled for all $(u,(x,y)) \in U_\rho \times \mathbb{R}^2, (x,y) \neq 0$. With Proposition 1.1.18 it follows, that all $(u,(x,y)) \in U_\rho \times \mathbb{R}^2, (x,y) \neq 0, u \in \operatorname{int} U_\rho$ are inner pairs.

Furthermore these $(u,(x,y))$ are also strong inner pairs: Note that for every $T > 0$and every $(u,(x,y))$ the pair $(u, \varphi(T,0,x,y,u))$ fulfills the controllability rank condition (5.5). Thus by Proposition 1.1.18 it follows, that $\varphi(T,0,x,y,u) \in \operatorname{int} \mathcal{O}^+(x,y)$ if $u \in \operatorname{int} U_\rho$. Because we can choose $T > 0$ arbitrary it follows, that $\varphi(t,0,x,y,u) \in \operatorname{int} \mathcal{O}^+(x,y)$ for all $t > 0$. In particular it follows, that for every $u \in \mathcal{U}$ which is piecewise constant and with $u(t) \in \operatorname{int} U$ for all $t \in \mathbb{R}$ we also have that for every $(x,y) \in \mathbb{R}^2, (x,y) \neq 0$ the pairs $(u,(x,y))$ are strong inner pairs.

Next we compute the Lyapunov spectrum of the system (5.1). If the projected system on $\mathbb{P}^1$ has two main control sets, then the Lyapunov spectrum consists of the eigenvalues of the linearized system for constant controls (cf. Theorem 1.2.12). Note that the Theorem 1.2.12 is only valid for systems in $\mathbb{R}^2$.

If we linearize the perturbed Duffing-van der Pol equation (5.1) at the singular point $x^* = (0,0)^T$ we get the bilinear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A_0 \begin{pmatrix} x \\ y \end{pmatrix} + u(t) A_1 \begin{pmatrix} x \\ y \end{pmatrix} \tag{5.6}$$
$$u \in \mathcal{U}_\rho = \{u : \mathbb{R} \to \mathbb{R} : u(t) \in U_\rho \text{ for a.a. } t \in \mathbb{R}, \text{ locally integrable}\}$$

with

$$A_0 := \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} \text{ and } A_1 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Note, that the eigenvalues $\lambda_{1,2}(u)$ for $u \in U_\rho$ of the right hand side of (5.6) are given by

$$\lambda_{1,2}(u) = \frac{1}{2}b \pm \frac{1}{2}\sqrt{b^2 + 4(a+u)}.$$

Instead of computing the projected system on the real projective space $\mathbb{P}^1$ it suffices to consider the projection of (5.6) onto the sphere $\mathbb{S}^1$. This is given by

$$\dot{s} = h_0(s) + u(t)h_1(s)$$

with

$$h_i(s) := [A_i + s^T A_i s \cdot \mathrm{id}]s.$$

We obtain

$$\begin{aligned} h_0(s_1, s_2) &= \begin{pmatrix} -s_1(s_1 s_2 + \alpha s_1 s_2 + \beta s_2^2) + s_2 \\ \alpha s_1 - s_2(s_1 s_2 + \alpha s_1 s_2 + \beta s_2^2) - \beta s_2) \end{pmatrix}, \\ h_1(s_1, s_2) &= \begin{pmatrix} -s_1^2 s_2 \\ s_1 - s_1 s_2^2 \end{pmatrix} \end{aligned}$$

and get

$$\begin{pmatrix} \dot{s}_1 \\ \dot{s}_2 \end{pmatrix} = \begin{pmatrix} -s_1(s_1 s_2 + \alpha s_1 s_2 + \beta s_2^2) + s_2 - s_1^2 s_2 \\ \alpha s_1 - s_2(s_1 s_2 + \alpha s_1 s_2 + \beta s_2^2) - \beta s_2 + s_1 - s_1 s_2^2 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}. \tag{5.7}$$

The sphere $\mathbb{S}^1$ can be parametrized by a curve $[0, 2\pi] \to \mathbb{S}^1, t \to (\cos(t), \sin(t))$. If we set $s(t) = (s_1(t), s_2(t)) = (\cos\delta(t), \sin\delta(t))$ we obtain

$$\begin{pmatrix} \dot{s}_1 \\ \dot{s}_2 \end{pmatrix} = \begin{pmatrix} \cos^2\delta(-\sin\delta - a\sin\delta + b - u(t)\sin\delta) - b\cos\delta + \sin\delta \\ (1 + a + u(t))\cos^3\delta + b\cos^2\delta\sin\delta - \cos\delta \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}.$$

By differentiating it follows that

$$\dot{s}(t) = \begin{pmatrix} -\sin\delta(t) \\ \cos\delta(t) \end{pmatrix} \dot{\delta}(t) = \begin{pmatrix} -s_2(t) \\ s_1(t) \end{pmatrix} \dot{\delta}(t).$$

Then by substitution we get the control system

$$\begin{aligned} &\dot{\delta} = (1 + a + u(t))\cos^2\delta \;\; + b\cos\delta\sin\delta =: g(\delta, u(t)) \\ &u \in \mathcal{U}_\rho = \{u : \mathbb{R} \to \mathbb{R} : u(t) \in U_\rho \text{ for a.a. } t \in \mathbb{R}, \text{ locally integrable}\}. \end{aligned} \tag{5.8}$$

on $[0, 2\pi)$.

Consider the set

$$M := \{\delta \in [0, \pi) : \text{there are } u_1, u_2 \in U_\rho \text{ with } g(\delta, u_1) < 0 \text{ and } g(\delta, u_2) > 0\}. \tag{5.9}$$

The main control sets of the projected system on $\mathbb{S}^1$ are represented by intervals $I \subset M$. For each $\delta \in I$ one can find a control value $u_1 \in U_\rho$ such that $g(\delta, u_1) > 0$ and a control value $u_2 \in U_\rho$ with $g(\delta, u_2) < 0$. Thus we can steer from every point of this interval to every other point. If M consists of two disjoint intervals, then the projected system (5.8) has two main control sets, and if it consists of only one interval, it has one main control set.

After these general results, we consider the perturbed Duffing-van der Pol equation (5.1) with

$$\alpha = -\frac{1}{4}, b = -2,$$
$$U_\rho = [-\rho, \rho] \text{ with } \rho^{\max} = \frac{6}{5}.$$

First we want to compute the Lyapunov spectrum of the associated bilinear system.

In Figure 5.1 we have computed for each $\delta \in [0, \pi]$ the set

$$\{g(\delta, u) : u \in U^\rho\}.$$

Thus we get, that for $\rho \in [0, \frac{3}{4})$ the set M defined as in (5.9) consists of two intervals, which means, that the projected system has two main control sets $D_{1,\rho}^{proj}, D_{2,\rho}^{proj}$. For $\rho \in (\frac{3}{4}, \frac{5}{4})$, the set M consists of only one interval, hence the projected system on $\mathbb{P}^1$ has only one main control set D_ρ^{proj}.

Since there are two main control sets for $\rho \in [0, \frac{3}{4})$, according to Theorem 1.2.12 we can calculate the Lyapunov spectrum by determining the eigenvalues of the right hand side of the bilinear control system (5.6) for *constant* controls, i.e.

$$\begin{aligned}\Sigma_{Ly} &= \Sigma_{Ly}(D_{1,\rho}^{proj}) \cup \Sigma_{Ly}(D_{2,\rho}^{proj}) \\ &= \left\{-1 \pm \frac{1}{2}\sqrt{3+4u} : u \in [-\rho, \rho]\right\} \\ &= [-1 - \frac{1}{2}\sqrt{3-4\rho}, -1 - \frac{1}{2}\sqrt{3+4\rho}] \cup [-1 + \frac{1}{2}\sqrt{3-4\rho}, -1 + \sqrt{3+4\rho}].\end{aligned}$$

We show that for the linearized control flow $\mathbf{T}\Phi$ on $\mathcal{U} \times \mathbb{R}^2$ there are two invariant subbundles $\mathcal{V}_1, \mathcal{V}_2$ with $\mathcal{U} \times \mathbb{R}^2 = \mathcal{V}_1 \oplus \mathcal{V}_2$ and

$$\Sigma_{Ly}(D_{1,\rho}^{proj}) = \Sigma_{Ly}(\mathcal{V}_{1,\rho}) \text{ and } \Sigma_{Ly}(D_{2,\rho}^{proj}) = \Sigma_{Ly}(\mathcal{V}_{2,\rho}). \tag{5.10}$$

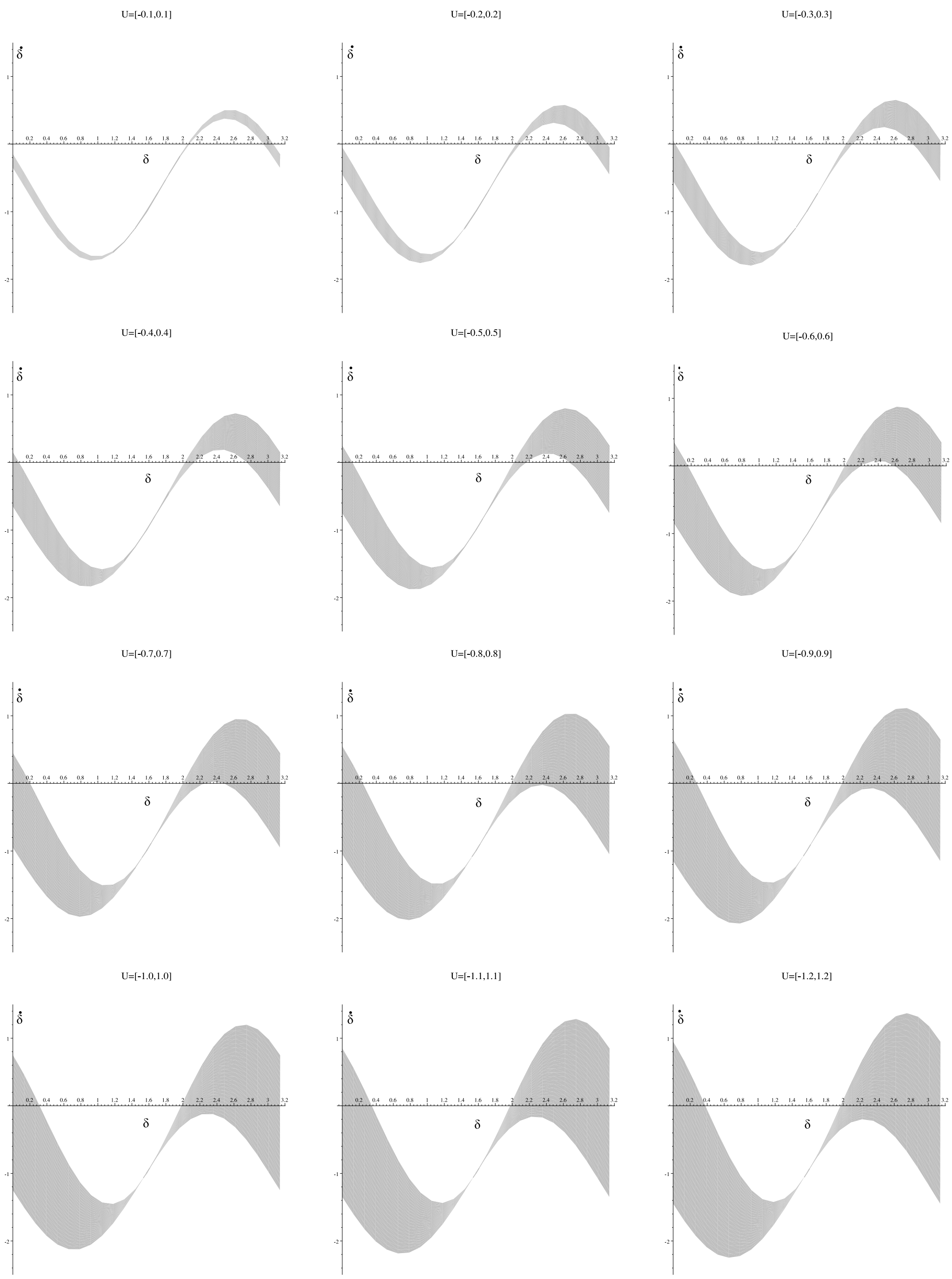

Figure 5.1: Control sets of the projected system.

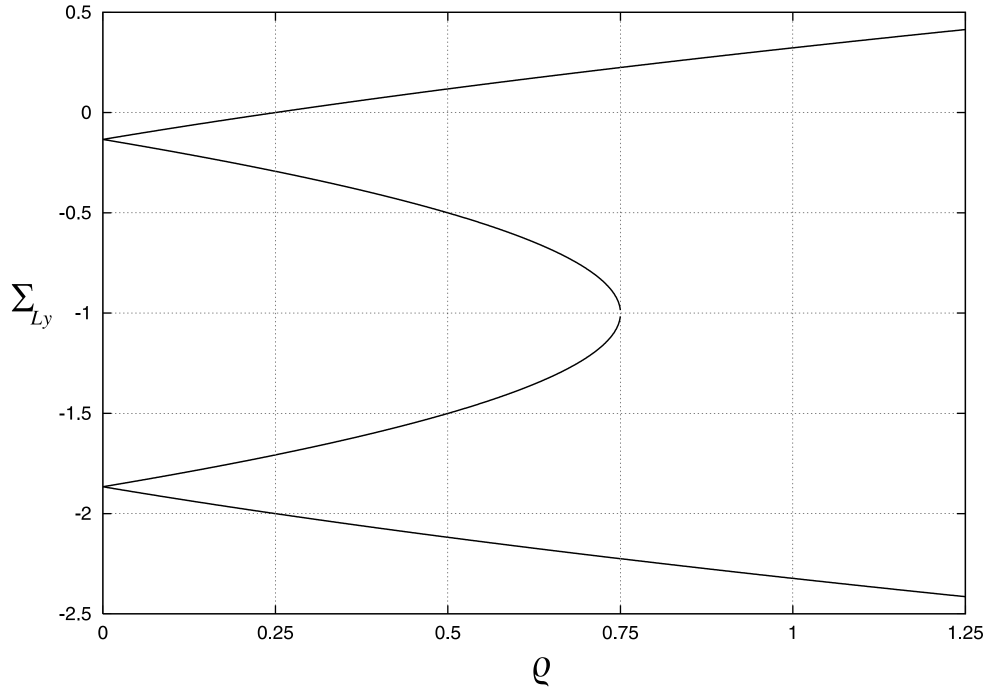

Figure 5.2: The Lyapunov spectrum of the Duffing-van der Pol oscillator for $\rho \in [0, \frac{3}{4})$ and its inner approximation for $\rho \in [\frac{3}{4}, \frac{5}{4}]$.

Note that the spectral intervals $\Sigma_{Ly}(D^{proj}_{1,\rho})$ and $\Sigma_{Ly}(D^{proj}_{2,\rho})$ do not overlap. Suppose, that there is one chain control set $E_\rho \subset \mathcal{U}\times\mathbb{P}^1$ with $D^{proj}_{1,\rho}, D^{proj}_{2,\rho} \subset E_\rho$. Then according to Theorem 1.2.12 it would follow, that $\Sigma_{Mo}(E_\rho) = \Sigma_{Ly}(D^{proj}_{1,\rho}) \cup \Sigma_{Ly}(D^{proj}_{2,\rho})$. But $\Sigma_{Mo}(E_\rho) = \Sigma_{Ly}$ must be a compact interval, which is a contradiction, because $\Sigma_{Ly}(D^{proj}_{1,\rho}) \cup \Sigma_{Ly}(D^{proj}_{2,\rho})$ is not an interval. Thus there are two chain control sets $E_{1,\rho}$ and $E_{2,\rho}$ with $D^{proj}_{i,\rho} \subset E^\rho_i, i = 1, 2$. The chain control sets $E_{1,\rho}$ and $E_{2,\rho}$ correspond to two subbundles $\mathcal{V}_{1,\rho}, \mathcal{V}_{2,\rho}$ with $\mathcal{U} \times \mathbb{R}^2 = \mathcal{V}_{1,\rho} \oplus \mathcal{V}_{2,\rho}$ and the relation (5.10).

In Figure 5.2 we have drawn the Lyapunov spectrum of the linearized system (5.6) for the interval $\rho \in [0, \frac{3}{4})$. We also have drawn the real parts of the eigenvalues for constant controls for $\rho \in [\frac{3}{4}, \frac{5}{4}]$. Note that these computed intervals may *not* be the Lyapunov spectrum Σ_{Ly}, but only an (possibly too small) approximation. Because for $\rho \in (\frac{3}{4}, \frac{5}{4}]$ the projected system (5.7) has only one main control set D^{proj}_ρ (cf. Figure 5.1), it is not enough to compute the eigenvalues of the linearized system for constant controls for determining the Lyapunov spectrum (cf. Theorem 1.2.12).

The numerical computations of the control sets in the Figure 5.3 and 5.4 where made with the program **CS** from Gerhard Häckl. A description of the underlying algorithm can be found in [16], [17], and in Appendix C of [9].

As we see in Figure 5.2, for $\rho \in [0, \frac{1}{4})$ the Lyapunov spectrum has only negative values. By Theorem 2.1.4 it follows, that there is a neighborhood around the singular point, in which there is no control set with nonvoid interior. Thus in the pictures in Figure 5.3 for $\rho = 0.1$ and $\rho = 0.2$ there are no control sets. In fact, for every constant $u \in U_\rho$ the corresponding differential equation is locally asymptotic stable at the origin. The grey cones in the picture are the eigenspaces of the linearized system (5.6), if we apply constant control functions $u \in U_\rho$. For every u we get two real eigenvalues. The eigenspace corresponding to the larger eigenvalue lies in the horizontal cone, and the eigenspace of the smaller eigenspace lies in the vertical cone. If one projects the vertical cones onto the sphere, then we get the control set $D^{proj}_{1,\rho}$ and if we project the horizontal cones onto the sphere we get the control set $D^{proj}_{2,\rho}$.

For $\rho = \frac{1}{4}$ we can not make a statement about the existence or nonexistence of control sets near the singular point. This comes from the fact, that the Lyapunov spectrum Σ_{Ly} has for this ρ-value the form $\Sigma_{Ly} = [-1-\frac{\sqrt{2}}{2}, -1] \cup [-1+\frac{\sqrt{2}}{2}, 0]$. We do not have the case where $\Sigma_{Ly} < 0$ or $0 \in \operatorname{int}\Sigma_{Ly}$, which we would need to be able to apply the Theorems in Chapter 2, 3 or 4. Numerical investigations let us believe that for this example, there is no control set with nonvoid interior and x^* in its closure. At the moment it is an open question, if in the case, where 0 is just the upper boundary of the Lyapunov spectrum of a nonlinear control system, there is a control set near the singular point or not.

For $\rho \in (\frac{1}{4}, \frac{3}{4})$, the perturbed Duffing-van der Pol oscillator and its spectrum actually fulfill the assumptions of Theorem 4.3.5, because we have $0 \in \operatorname{int}\Sigma_{Ly}$. Thus we can conclude, that there are two control sets $D_{>,\rho}$ and $D_{<,\rho}$ with nonvoid interior and $x^* \in \operatorname{cl} D_{>,\rho} \cap \operatorname{cl} D_{<,\rho}$. In Figure 5.3 we have computed these control sets, which are the small leafs emerging from the origin. Note, that here in this example $D_{>,\rho}$ and $D_{<,\rho}$ do not coincide. Coincidence of the control sets $D_{<,\rho}$ and $D_{>,\rho}$ does not follow

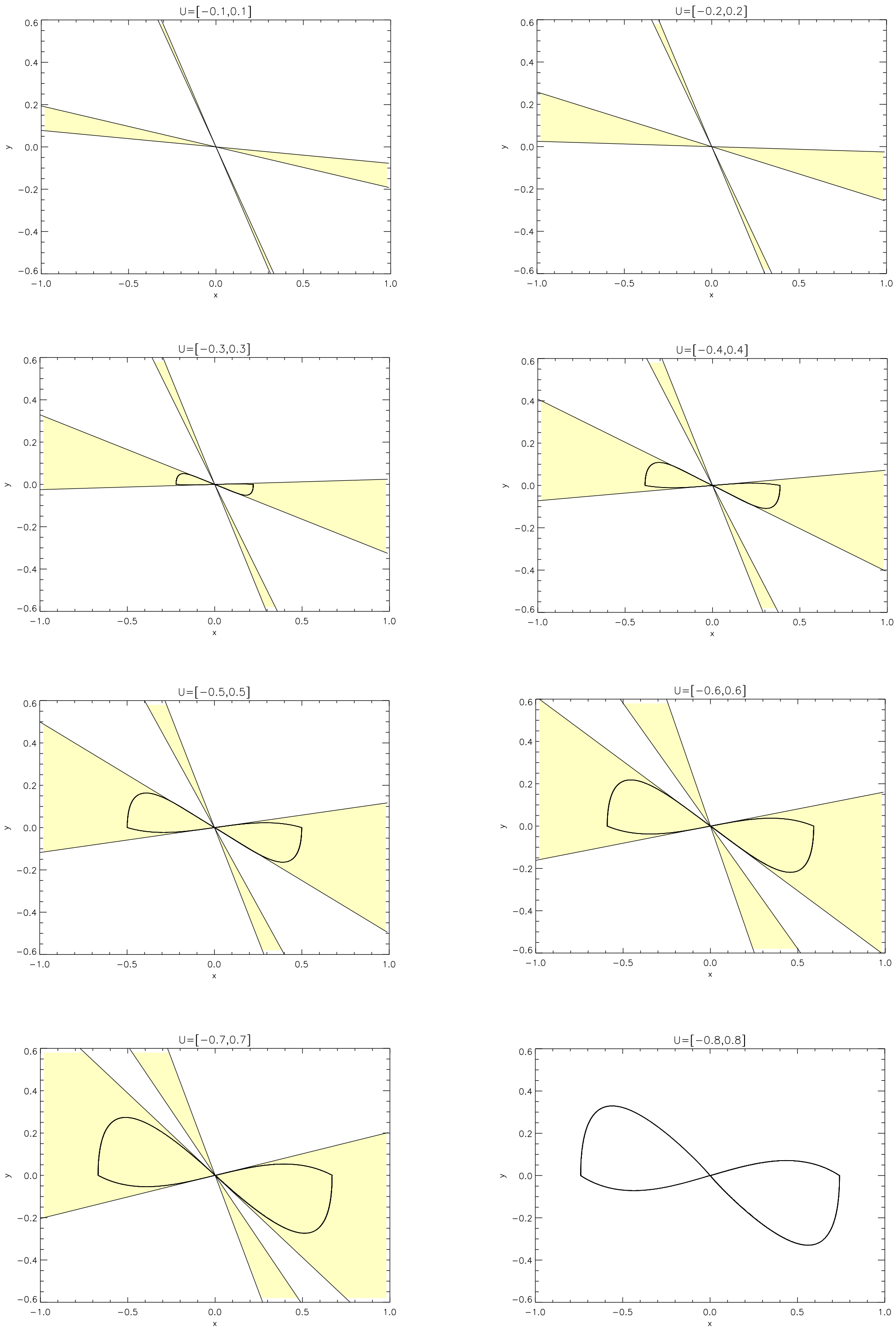

Figure 5.3: The control sets of the perturbed Duffing-van der Pol oscillator.

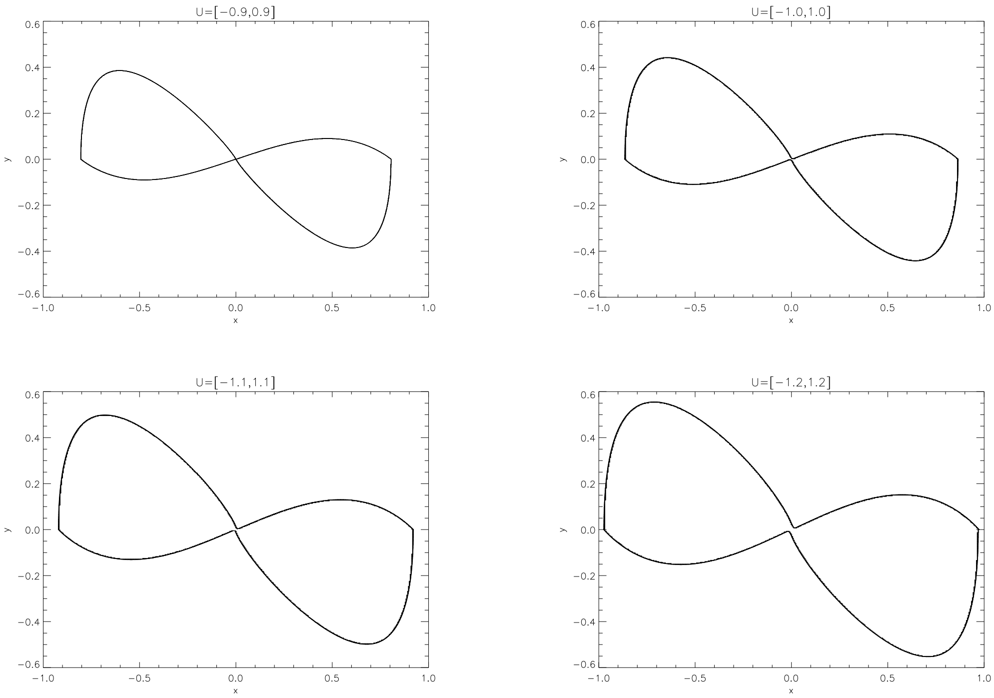

Figure 5.4: The control sets of the perturbed Duffing-van der Pol oscillator.

from Theorem 4.3.5 or Theorem 4.2.5, because these Theorems make only local results around the singular point x^*, and give us the existence of the two control sets.

The grey cones are the eigenspaces of A_0+uA_1 for $u \in (-\rho, \rho)$. The horizontal cones which intersect the control sets $D_{>,\rho}$ and $D_{<,\rho}$ correspond to eigenvalues of the spectral interval $\Sigma_{Ly}(\mathcal{V}_{2,\rho})$ with $0 \in \operatorname{int}\Sigma_{Ly}(\mathcal{V}_{2,\rho})$. These cones are in fact control sets for the bilinear system (5.6), see Corollary 4.3.1. Their projection on the projective space $\mathbb{P}^1$ form the control set $D_{2,\rho}^{proj}$ of the projected control system (5.7). Thus the pictures show, that here the control sets of the linearized system look locally like the control sets of the nonlinear system. As proven in Theorem 4.3.5, the local unstable manifolds corresponding to positive Lyapunov exponents lie in the interior of the control sets.

The other two vertical cones correspond to the eigenvalues of the spectral interval $\Sigma_{Ly}(\mathcal{V}_{1,\rho})$ which has only negative values, i.e. $\Sigma_{Ly}(\mathcal{V}_{1,\rho}) \subset \mathbb{R}^-$. The bilinear control system (5.6) is not controllable in these cones according to Corollary 4.3.1. The projection of these cones onto the real projective space $\mathbb{P}^1$ form the control sets $D_{1,\rho}^{proj}$ for the projected system (5.7).

For $\rho = \frac{3}{4}$ it is not clear, if there are still two main control sets for the projected system, or if these two control sets are melt together to form a new control set. For $u = \frac{3}{4}$, the right hand side A_0+uA_1 has the eigenvalue -1 with corresponding eigenspace $E(-1) = \mathbb{R}^2$. Thus we now do not draw the grey cones for the eigenspaces of the linearized system (5.6), because they would fill the whole plane. The projected control system (5.7) has only one chain control set E_ρ with $E_\rho = \mathbb{R}^2$.

Now for $\rho \in (\frac{3}{4}, \frac{6}{5}]$, we see in Figure 5.1, that there is only one control set $D_\rho^{proj} \subset \mathbb{P}^1$ for the projected control system (5.7). By Theorem 1.2.12 the eigenvalues of the right hand side of the bilinear system for constant $u \in U_\rho$ are just an approximation of the Lyapunov spectrum. Thus for $\rho \in (\frac{3}{4}, \frac{5}{4}]$ in Figure 5.2 we have not drawn the Lyapunov spectrum but only an inner approximation. In the Figures 5.4 we have not drawn the grey cones anymore. There are control values $u \in U_\rho$ for which the matrices A_0+uA_1 have complex eigenvalues. Thus the corresponding generalized real eigenspaces are $\mathbb{R}^2$.

Because there exist control sets with nonvoid interior for $\rho \in (\frac{1}{4}, \frac{3}{4})$ it follows, that for $\rho \in [\frac{3}{4}, \frac{5}{4})$ we have control sets $D_{>,\rho}$ and $D_{<,\rho}$ with nonvoid interior. Thus in this case, we do not have to use the Theorem 4.3.5 to get the existence of the control sets because the control set $D_{>,\rho'}$ and $D_{<,\rho'}$ for $\rho' \in (\frac{1}{4}, \frac{3}{4})$ are included in them. But their existence would also follow from this Theorem. For $\rho > 0.9$ one investigates, that the control sets $D_{>,\rho}$ and $D_{<,\rho}$ are melt together to one control set D_ρ. Numerical investigations for $\rho \in [\frac{3}{4}, 0.9]$ did not show, if the control sets $D_{>,\rho}$ and $D_{<,\rho}$ coincide, i.e. if the two leafs melt together. Here the numerical algorithms (and the Theorems in this book) are not accurate enough.

5.2 The Lorenz Equation

The Lorenz equation serves as a finite dimensional model of the Rayleigh-Benard convection [28]. It is a classical example of a system with complex dynamics and is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \sigma(y-x) \\ \rho x - y - xz \\ -\beta z + xy \end{pmatrix}. \tag{5.11}$$

with parameters $\sigma, \rho, \beta > 0$. For a time variant perturbation in ρ the system was examined in [9] Chapter 13.2. Here we consider a time variant perturbation of the system (5.11)

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} &= f_0(x,y,z) + \textstyle\sum_{i=1}^{3} u_i(t) f_i(x,y,z) \\ u = (u_1, u_{2,}, u_3) \in \mathcal{U} &= \{u : \mathbb{R} \to \mathbb{R}^3 : u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{ locally integrable}\} \end{aligned} \tag{5.12}$$

with a compact subset $U \subset \mathbb{R}^2$ with nonvoid interior and

$$f_0(x,y,z) := \begin{pmatrix} \sigma(y-x) \\ \rho x - y - xz \\ -\beta z + xy \end{pmatrix} \quad f_1(x,y,z) := \begin{pmatrix} 0 \\ x - y + z \\ -z \end{pmatrix}$$
$$f_2(x,y,z) := \begin{pmatrix} z \\ 0 \\ x + y - z \end{pmatrix} \quad f_3(x,y,z) := \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}.$$

The system was constructed in such a way, that we can apply the existence Theorem 4.1.12 and verify the results numerically.

This system has the singular point

$$x^* = (0,0,0)^T \in \mathbb{R}^3.$$

We will show that the perturbed Lorenz system (5.12) has for certain parameters ρ, σ, β and a certain control range U a control set $D \subset \mathbb{R}^3$ with nonvoid interior and $x^* \in \operatorname{cl} D$ by applying Theorem 3.7.1.

Before we can apply this Theorem, we have to verify the assumptions of this Theorem. First we show, that the system (5.12) is locally accessible on $\mathbb{R}^3 \setminus \{0\}$ for all $\sigma, \rho, \beta > 0$. This means, we have to check if the accessibility rank condition

$$\dim \operatorname{span}\{\operatorname{ad}_{f_i}^k f_j : k \in \mathbb{N}_0 \text{ and } i, j \in \{0,1,2,3\}\}(x,y,z) = 3 \tag{5.13}$$

for all $(x,y,z) \in \mathbb{R}^3 \setminus \{0\}$ is fulfilled. It turns out, that

$$\dim \operatorname{span}\{f_0, f_1, f_2, f_3, \operatorname{ad}_{f_0} f_1, \operatorname{ad}_{f_1} f_2\}(x,y,z) = 3 \text{ for all } (x,y,z) \in \mathbb{R}^3 \setminus \{0\}. \tag{5.14}$$

To verify this we have computed the following Lie brackets

$$\mathrm{ad}_{f_0} f_1(x,y,z) = \begin{pmatrix} -\sigma(x-y+z) \\ \sigma(y-x)+x(1-\rho)+xy+z(1-\beta)+xy \\ -x(x-y+z) \end{pmatrix},$$

$$\mathrm{ad}_{f_1} f_2(x,y,z) = \begin{pmatrix} 0 \\ -x-y \\ 0 \end{pmatrix}.$$

In the following we have written down the vector fields for the different points in the state space, which one needs to verify (5.14).

- For $(x,0,0)^T \in \mathbb{R}^3, x \neq 0$

$$\dim \operatorname{span}\{f_0, f_1, f_2\}\,(x,0,0) = 3.$$

- For $(0,y,0)^T \in \mathbb{R}^3, y \neq 0$

$$\dim \operatorname{span}\{f_0, f_1, f_2\}\,(0,y,0) = 3.$$

- For $(0,0,z)^T \in \mathbb{R}^3, z \neq 0$

$$\dim \operatorname{span}\{f_1, f_2, f_3\}\,(0,0,z) = 3.$$

- For $(x,y,0)^T \in \mathbb{R}^3, (x,y) \neq (0,0)$

$$\dim \operatorname{span}\{f_1, f_2, \mathrm{ad}_{f_0} f_2, \mathrm{ad}_{f_1} f_2\}\,(x,y,0) = 3.$$

- For $(x,0,z)^T \in \mathbb{R}^3, (x,z) \neq (0,0)$

$$\dim \operatorname{span}\{f_1, f_2, f_3, \mathrm{ad}_{f_1} f_2\}\,(x,0,z) = 3.$$

- For $(0,y,z)^T \in \mathbb{R}^3, (y,z) \neq (0,0)$

$$\dim \operatorname{span}\{f_0, f_1, f_2, f_3\}\,(0,y,z) = 3.$$

- For $(x,y,z)^T \in \mathbb{R}^3, (x,y,z) \neq (0,0)$

$$\dim \operatorname{span}\{f_1, f_2, f_3, \mathrm{ad}_{f_1} f_2\}\,(x,y,z) = 3.$$

Now we choose

$$\sigma = 4, \beta = \frac{8}{3}, \rho = \frac{3}{4} \text{ and}$$
$$U = [-0.8, 0.8] \times [-0.6, 0, 6] \times [-0.2, 0.2] \subset \mathbb{R}^3.$$

We will show, that for all $u \in \operatorname{int} U$ and all $(x,y,z)^T \in \mathbb{R}^3 \setminus \{0\}$ the pairs $(u,(x,y,z))$ are inner pairs. For $u \in U$ we define

$$f(x,y,z,u) := f_0(x,y,z) + \sum_{i=1}^{3} u_i(t) f_i(x,y,z).$$

By Proposition 1.1.18 we have to check if the controllability rank condition

$$\dim \operatorname{span}\{(f, ad_f^k f_i) : i = 1, \dots, 3, k = 0, 1, \dots\}(x,y,z,u) = 3 \tag{5.15}$$

is fulfilled. It turns out that

$$\dim \operatorname{span}\{f, f_1, f_2, f_3, \operatorname{ad}_f f_2, \operatorname{ad}_f f_3\}(x,y,z,u) = 3$$

for every $(x,y,z)^T \in \mathbb{R}^3 \setminus \{0\}$ and every $u \in U$. For verifying this we have computed the following Lie brackets with Maple

$$\operatorname{ad}_f f_2(x,y,z) = \begin{pmatrix} xy + z(\sigma - \beta - u_3) \\ x(x + y - z - u_1) - u_1 y + z(z - \rho) \\ x(-y - z + \beta + \rho - \sigma + u_1 + u_3) + y(-1 - z + -u_1 + \beta + u_3) + u_1 z \end{pmatrix}$$

$$\operatorname{ad}_f f_3(x,y,z) = \begin{pmatrix} u_2 z \\ -z(x - u_1) \\ -x(y + u_2) - u_2 y \end{pmatrix}$$

As above we have to consider the following cases.

- For $(x,0,0)^T \in \mathbb{R}^3, x \neq 0$

$$\dim \operatorname{span}\{f, f_1, f_2\}\,(x,0,0,u) = 3.$$

- For $(0,y,0)^T \in \mathbb{R}^3, y \neq 0$

$$\dim \operatorname{span}\{f, f_1, f_2\}\,(0,y,0,u) = 3.$$

- For $(0,0,z)^T \in \mathbb{R}^3, z \neq 0$

$$\dim \operatorname{span}\{f_1, f_2, f_3\}\,(0,0,z,u) = 3.$$

- For $(x,y,0)^T \in \mathbb{R}^3, (x,y) \neq (0,0)$

$$\dim \operatorname{span}\{f, f_1, \operatorname{ad}_f f_2, \operatorname{ad}_f f_3\}\,(x,y,0,u) = 3.$$

- For $(x,0,z)^T \in \mathbb{R}^3, (x,z) \neq (0,0)$

$$\dim \operatorname{span}\{f, f_1, \operatorname{ad}_f f_2, \operatorname{ad}_f f_3\}\,(x,0,z,u) = 3.$$

- For $(0,y,z)^T \in \mathbb{R}^3, (y,z) \neq (0,0)$

$$\dim \operatorname{span}\{f, f_1, f_2, f_3, \operatorname{ad}_f f_3\}\,(0,y,z,u) = 3.$$

- For $(x,y,z)^T \in \mathbb{R}^3, (x,y,z) \neq (0,0,0)$

$$\dim \operatorname{span}\{f, f_1, f_2, f_3, \operatorname{ad}_f f_2, \operatorname{ad}_f f_3\}\,(x,y,z,u) = 3.$$

Thus the controllability rank condition (5.5) holds for all $(u,(x,y,z)^T) \in U \times \mathbb{R}^3$ with $(x,y,z) \neq 0$. With Proposition 1.1.18 it follows, that for all $(x,y,z) \neq 0$ and $u \in \operatorname{int} U$ the pairs $(u,(x,y,z)^T) \in U \times \mathbb{R}^3$ are inner pairs and therefore $(u,(x,y,y)^T)$ are also strong inner pairs.

By linearizing the nonlinear control system (5.12) at the singular point x^* we get the bilinear control system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & u_2 \\ \rho + u_1 & u_1 - 1 & u_1 \\ u_2 & u_2 & -\beta - u_2 - u_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^3 : u(t) \in U \text{ for a.a. } t \in \mathbb{R}, \text{ locally integrable}\} \tag{5.16}$$

For getting the existence of a control set $D \subset \mathbb{R}^3$ with nonvoid interior by Theorem 4.1.12, we have to find two periodic control functions $u^h, u^s \in \mathcal{U}$ such that the corresponding Lyapunov exponents fulfill

$$\begin{array}{rl} 0 & > \lambda_1^s \geq \lambda_2^s \geq \lambda_3^s \quad \text{and} \\ \lambda_1^h > \; 0 & > \lambda_1^h \geq \lambda_3^h. \end{array} \tag{5.17}$$

To make things easier, we consider only constant control functions. Here it is not necessary to calculate the exact Lyapunov spectrum. The matrix on the right hand side of (5.16) has three eigenvalues $\lambda_i(u) \in \mathbb{R}, i = 1,2,3$ for $u \in U$. By numerical computation with the program **Maple** we get the approximative results

$$\begin{aligned} \{\lambda_1(u) : u \in U\} &\approx [-0.6536, +0.1523], \\ \{\lambda_2(u) : u \in U\} &\approx [-3.7501, -1.8899], \\ \{\lambda_3(u) : u \in U\} &\approx [-5.5918, -3.8240]. \end{aligned}$$

by discretizing the control range U. Since $\{\lambda_i(u) : u \in U\} \cap \{\lambda_j(u) : u \in U\} = \emptyset$ for $i \neq j$, the eigenspaces $E(\lambda_i(u))$ are onedimensional. Since $0 \in \operatorname{int}\{\operatorname{Re}\sigma_1(u) : u \in U\}$, there is a constant control function $u^h \in U$ such that $\lambda_1^h > 0$ and $\lambda_2^h, \lambda_3^h < 0$. There is also a control function u^s with $\lambda_i^s < 0$. By Theorem 4.1.12 we conclude, that the perturbed Lorenz system (5.12) has two control sets $D_<$ and $D_>$ with nonvoid interior and $x^* \in \operatorname{cl} D_> \cap \operatorname{cl} D_<$.

Because we have not computed the Lyapunov spectrum, we can not say, if there are subbundles $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{U} \times \mathbb{R}^d$ with $\mathcal{V}_1 \oplus \mathcal{V}_2$ such that $\Sigma_{Ly}(\mathcal{V}_1) \cap \Sigma_{Ly}(\mathcal{V}_2) = \emptyset$ and

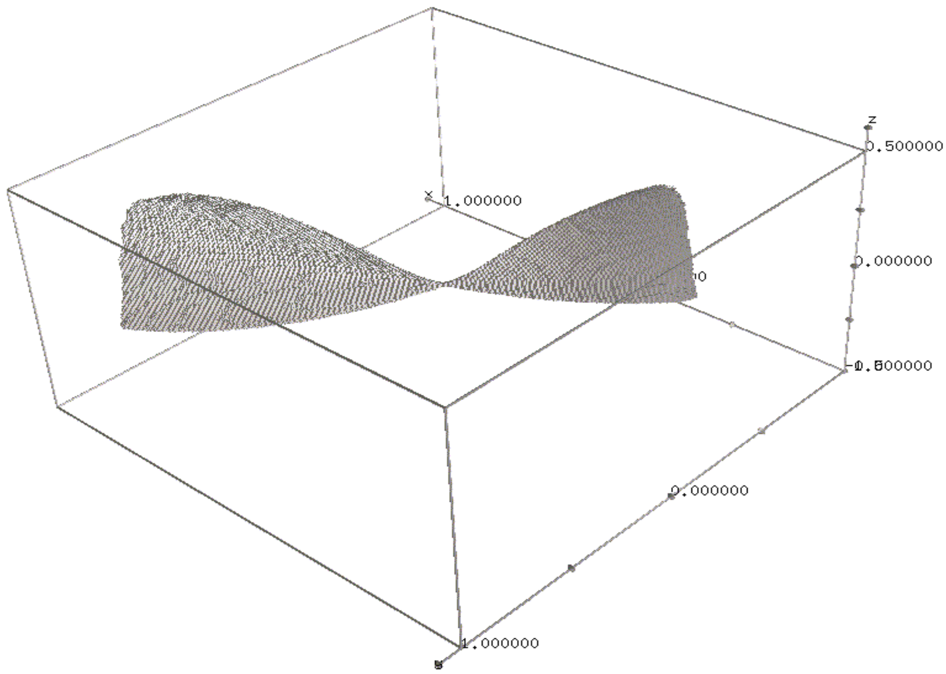

Figure 5.5: The control sets of the perturbed Lorenz equation.

$0 \in \operatorname{int} \Sigma_{Ly}(\mathcal{V}_2), \Sigma_{Ly}(\mathcal{V}_1) \subset \mathbb{R}^-$. Thus we can not apply the Theorem 4.3.5 which would characterize the local behavior of the control sets in relation to the bilinear control system.

We verified the theoretical result with the program **gaio** by M.Dellnitz, A.Hohmann and O.Junge. **gaio** is a program for computing unstable manifolds and global attractors of a dynamical system (cf. [10] and [11]). The program and the algorithm was modified by D.Szolnoki to compute the control sets of a control system, (cf. [31] and [30]). The result can be seen in Figure 5.5 and Figure 5.6.

In Figure 5.5 and Figure 5.6 we discern two wings emerging from the origin, the control sets $D_<$ and $D_>$. The numerical experiments let us believe, that $D_>$ and $D_<$ are disjoint control sets. But this is (as it was in the case of the Duffing-van der Pol oscillator) not proven. Here are the limits of the local analysis of control systems by linearizing at the singular point.

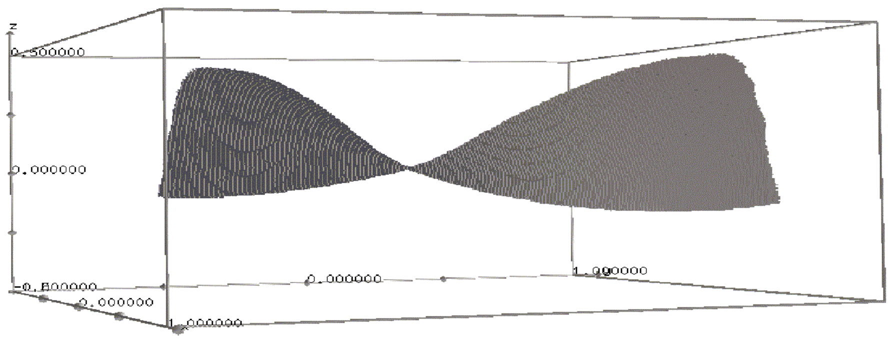

Figure 5.6: Another view of the control sets of the Lorenz equation.

Chapter 6

Invariant Fibre Bundles

In this Appendix, we will summarize the results of the qualitative theory of ordinary differential equations, which we need in this book.

Since the times of Poincaré, the geometric structure of the phase space of *autonomous* differential equations was investigated. Connected with this investigations are the famous Hartman-Grobman theorem and the decoupling of differential equations.

But if we look at a control system and fix a (time variant) control function, then this results in a *nonautonomous* differential equation. Hence the classical theory for autonomous differential equations does not apply. However in the last years there where attempts to overcome this problem. Here the works of B.Aulbach and T.Wanner (see for example [6]) are to mention and the work of S.Siegmund [29] on which this chapter is based.

The first part of this Appendix cites the results of [29] which are relevant in this thesis and gives us a short introduction into the theory of invariant fibre bundles. We start by looking at linear differential equations. After having defined the term of the linear integral manifold, we come to the dichotomy spectrum of a linear system. This plays a crucial part for the rest of this Appendix. Then we characterize the dichotomy spectrum of the linear differential equation by the linear integral manifolds (Proposition 6.1.7). The linear integral manifolds can be described by the long time behavior of trajectories, which itself can be expressed by the term of quasiboundedness.

An interesting question is, if the dichotomy spectrum stays invariant, if we transform the linear differential equation. The result is, that under a so called kinematic transformation the spectrum stays invariant (Proposition 6.1.9).

Next we come to the nonlinear theory. We consider a class of differential equation of the form $\dot{x} = A(t)x + F(t,x)$ with fixed point 0. Invariant fibre bundles (or integral manifolds) are the generalizations of the stable and unstable manifolds in $\mathbb{R}^d$ of autonomous systems. Because for trajectories of nonautonomous systems the starting time is also important, the invariant fibre bundles lie in the extended phase space $\mathbb{R} \times \mathbb{R}^d$. In Theorem 6.1.13 we get a first existence result for invariant fibre bundles through 0 and a characterization by quasiboundedness properties of the trajectories, starting in them. Intersection of invariant fibre bundles are also invariant fibre bundles (Theorem 6.1.16) and we get invariant fibre bundles through every point in the extended phase space (Theorem 6.1.18).

By asymptotic phases one can project trajectories on an invariant fibre bundle. We compare the original with the projected trajectory. Finally a Hartman-Grobman Theorem for nonautonomous systems is stated (Theorem 6.1.22).

The second part of this appendix concentrates on the local theory of a special class of periodic systems. These systems appear for example, if we apply a periodic control functions to a control affine system.

Now the global results of the first part of this Appendix were only possible, because we assumed that the right hand side of the differential equation fulfills certain quantitative properties. In general, these properties are not fulfilled. But we can restrict our given system by a cut-off technique, such that the restricted system fulfills the assumptions of the global theory. Hence for the time periodic right hand side we first apply a Floquet transformation, such that the linear part gets autonomous. Then by restricting this system by a radial retraction, we can apply the global theory to the restricted system. The integral manifolds of the restricted system can now be used to define a local equivalent for the given nonlinear system. We investigate, how these objects are interrelated, and compare the linear with the nonlinear fibre bundles.

6.1 Global Theory

6.1.1 Linear Theory

Because our aim is to characterize nonlinear differentials equation via invariant fibre bundles, we first have to understand linear systems. After introducing the concept of linear manifolds for a linear system, we come to the term of exponential dichotomy and the dichotomy spectrum. It will be shown, that the spectrum is invariant, if we transform the linear system by a so called kinematic transformation.

We consider the linear differential equation on $\mathbb{R}^d$

$$\dot{x} = A(t)x \tag{6.1}$$

where $A : \mathbb{R} \rightarrow \mathcal{L}(\mathbb{R}^d)$ is a locally integrable function, with fundamental solution $\eta(t, \tau)$, $t, \tau \in \mathbb{R}$.

Definition 6.1.1 *A nonempty set $\mathcal{W} \subset \mathbb{R} \times \mathbb{R}^d$ is called* linear integral manifold *of (6.1), if it fulfills the following properties.*

(a) $\mathcal{W}$ is invariant *for (6.1), i.e. we have*

$$(\tau, \eta(t, \tau)x) \in \mathcal{W}$$

for all $(\tau, x) \in \mathcal{W}$ and all $t \in \mathbb{R}$.

(b) For every $\tau \in \mathbb{R}$ the fibre

$$\mathcal{W}(\tau) = \{x \in \mathbb{R}^d : (\tau, x) \in \mathcal{W}\}$$

is a linear subspace of $\mathbb{R}^d$.

Linear integral manifolds are in fact *topological* manifolds in $\mathbb{R}^{1+d}$ and vector bundles over $\mathbb{R}$, see Lemma 2.19 in [29]. Thus the *fibre dimension* and the dimension of the linear integral manifolds can be defined as $\dim \mathcal{W} = \dim \mathcal{W}(\tau)$ for $\tau \in \mathbb{R}$.

The idea of exponential dichotomy is based on considerations on autonomous linear differential equations $\dot{x} = Ax$. If A has eigenvalues with positive and negative real parts, then solutions can be split into two components. The first component goes in positive time to zero and the second in negative time to zero. A generalization of this concept for nonautonomous linear systems is the term of the exponential dichotomy.

Definition 6.1.2 *The differential equation (6.1) has an* exponential dichotomy *if there exist constants $K \geq 1, \alpha > 0$ and an* invariant projector $P : \mathbb{R} \to \mathcal{L}(\mathbb{R}^d)$, *satisfying*

$$\begin{aligned} \|\eta(t,\tau)P(\tau)\| &\leq K e^{-\alpha(t-\tau)} \quad \text{for } t \geq \tau, \\ \|\eta(t,\tau)[\mathrm{id} - P(\tau)]\| &\leq K e^{\alpha(t-\tau)} \quad \text{for } \tau \geq t. \end{aligned}$$

Here an invariant projector $P : \mathbb{R} \to \mathcal{L}(\mathbb{R}^d)$ is a mapping which consists of projections $P(t) \in \mathcal{L}(\mathbb{R}^d)$ for all $t \in \mathbb{R}$ and with

$$P(t)\eta(t,\tau) = \eta(t,\tau)P(\tau) \text{ for all } t, \tau \in \mathbb{R}.$$

Definition 6.1.3 *Consider the linear differential equation (6.1). Then the set*

$$\Sigma(A) = \{\gamma \in \mathbb{R} : \dot{x} = [A(t) - \gamma \cdot \mathrm{id}]x \text{ has no exponential dichotomy}\}$$

is called the dichotomy spectrum *of (6.1). If there is a constant $a \geq 0$ with $\Sigma(A) \subset [-a, a]$, then the spectrum $\Sigma(A)$ is called* bounded.

Remark 6.1.4 *Note that we defined here the dichotomy spectrum $\Sigma(A)$ for* one *differential equation $\dot{x} = A(t)x$. The generalization of this spectrum for bilinear control systems is given in (2.2.2) by the dichotomy spectrum Σ_{dich}.*

The notion of quasi boundedness goes back to B.Aulbach [5]. It describes the exponential growth of functions, and helps us to characterize linear integral manifolds of the linear system (6.1) and the nonlinear integral manifolds later in this chapter.

Definition 6.1.5 *Let $d \in \mathbb{N}, \gamma \in \mathbb{R}$ and $I \subset \mathbb{R}$ be a nonvoid interval, and let $g : I \to \mathbb{R}^d$ be a measurable function. Then the mapping g is called*

(a) γ^+-quasibounded *for $t \to \infty$, if I is unbounded to the right and the inequality*

$$\operatorname{ess\,sup}\{\|g(t)\| \, e^{-\gamma t} : t \geq \tau\} < \infty$$

is satisfied for a $\tau \in I$.

(b) γ^--quasibounded *for* $t \to -\infty$, *if* I *is unbounded to the left and the inequality*

$$\operatorname{ess\,sup}\{\|g(t)\|\, e^{-\gamma t} : t \leq \tau\} < \infty$$

is satisfied for a $\tau \in I$.

(c) $\gamma^{\pm}$-quasibounded *if* $I = \mathbb{R}$ *and the inequality*

$$\operatorname{ess\,sup}\{\|g(t)\|\, e^{-\gamma t} : t \in \mathbb{R}\} < \infty$$

is satisfied.

The number γ *is called* exponential growth rate .

To characterize the dichotomy spectrum of a linear differential equation, we have to introduce the notion of *bounded growth.*

Definition 6.1.6 *The linear differential equation (6.1) has* bounded growth *if there are constants* $K \geq 1$ *and* $\alpha \geq 0$ *with*

$$\|\eta(t,\tau)\| \leq K e^{a|t-\tau|} \text{ for all } t, \tau \in \mathbb{R}.$$

In [29], Satz 2.42 one can find criterions on A, under which the linear differential equation has bounded growth.

The following proposition shows the equivalence between the bounded growth of a linear differential equation and the decomposition of the spectral interval into a finite number of compact intervals. Associated to each interval there is a linear integral manifold.

Proposition 6.1.7 *Consider the linear differential equation (6.1). Then the following statements are equivalent.*

(a) *The linear differential equation (6.1) has bounded growth.*

(b) *The linear system (6.1) has a compact nonempty spectrum*

$$\Sigma(A) = [a_1, b_1] \cup \ldots \cup [a_n, b_n], \text{ with } 1 \leq n \leq d$$

and $-\infty < a_1 \leq b_1 < a_2 \leq b_2 < \ldots < a_n \leq b_n < \infty$. *Corresponding to the spectral intervals, there exist the so called* spectral manifolds $\mathcal{W}_1, \ldots, \mathcal{W}_n$ *which are linear integral manifolds with* $\mathcal{W}_i \cap \mathcal{W}_j = \mathbb{R} \times \{0\}$ *and can be characterized in the following way*

$$\mathcal{W}_i = \left\{ (\tau, x) \in \mathbb{R} \times \mathbb{R}^d : \begin{array}{c} \eta(\cdot,\tau)x \text{ is } \gamma^+\text{-quaisbounded for all } \gamma \in (b_i, a_{i+1}) \\ \text{and } \delta^-\text{-quasibounded for all } \delta \in (b_{i-1}, a_i). \end{array} \right\},$$

such that we have $\mathcal{W}_1 \oplus \ldots \oplus \mathcal{W}_n = \mathbb{R} \times \mathbb{R}^d$. *Here* $\mathcal{W}_1 \oplus \ldots \oplus \mathcal{W}_n$ *denotes the Whitney sum of the vector bundles* $\mathcal{W}_1, \ldots, \mathcal{W}_n$.

Proof. This follows from Satz 2.37 in Siegmund [29]. ∎

If our linear differential equation (6.1) is periodic, one can find a transformation, which transforms the system into an autonomous linear differential equation by Floquet Theory, see for example G.Sansone and R.Conti [27]. This is done for example in Section 6.2.1. The question is, if such a transformation changes the dichotomy spectrum. The kinematic similarity transformations are a class of transformations which do not change the dichotomy spectrum

Definition 6.1.8 *Consider two linear differential equation*

$$\dot{x} = A(t)x \tag{6.2}$$

and

$$\dot{x} = B(t)x \tag{6.3}$$

with locally integrable functions $A, B : \mathbb{R} \to \mathcal{L}(\mathbb{R}^d)$. Then the differential equations (6.2) and (6.3) are called kinematically similar *, if there is a solution $S : \mathbb{R} \to gl(\mathbb{R}^d)$ of the matrix differential equation*

$$\dot{S} = A(t)S - SB(t)$$

on $\mathbb{R}$, which is together with its inverse $\mathbb{R} \to gl(\mathbb{R}^d), t \mapsto S^{-1}(t)$ bounded on $\mathbb{R}$. Then $A(t)$ is called kinematically similar to $B(t)$. *The mapping $(t, x) \mapsto S(t)x$ is called* kinematic similarity transformation .

The Floquet transformation, which transforms a linear periodic differential equation into a linear autonomous differential equation is an example for a kinematic similarity transformation. We get the following result.

Proposition 6.1.9 *Consider the linear differential equation*

$$\dot{x} = A(t)x \tag{6.4}$$

with locally integrable function $A : \mathbb{R} \to \mathcal{L}(\mathbb{R}^d)$ and bounded growth. Assume that for a $\gamma \in \mathbb{R}$ the linear differential equation $\dot{x} = [A(t) - \gamma\,\mathrm{id}]x$ has an exponential dichotomy with constants $K \geq 1, \alpha > 0$ and invariant projector P. Denote by $[a_i, b_i], i = 1, \dots, n$ the spectral intervals and $\mathcal{W}_i$ the associated spectral manifolds with fibre dimension $d_i = \dim \mathcal{W}_i, d_1 + \ldots + d_n = d$, which are given by Proposition 6.1.7. Then (6.4) is kinematically similar to a decoupled linear differential equation

$$\dot{x} = \begin{pmatrix} B_1(t) & & \\ & \ddots & \\ & & B_n(t) \end{pmatrix} x$$

with locally integrable functions $B_i : \mathbb{R} \to \mathcal{L}(\mathbb{R}^{d_i}), i = 1, \dots, n$ *and bounded growth. The linear differential equations* $\dot{x} = B_i(t)x$ *have the spectrum* $[a_i, b_i]$. *The corresponding linear kinematic transformation* S *has the properties*

$$\|S(t)\| \le \left(\sqrt{2}\right)^{n-1} \quad \text{and} \quad \left\|S^{-1}(t)\right\| \le \left(\sqrt{2}K\right)^{n-1} \quad \text{for all } t \in \mathbb{R}.$$

Proof. This follows from Satz 3.26 and Satz 3.28 in Siegmund [29]. ■

6.1.2 Invariant Fibre Bundles

Now we come to the (global) existence theorem for invariant fibre bundles. Here we consider a class of nonlinear systems, which we get for example by linearizing a differential equation at a fixed point.

Definition 6.1.10 *Let* $A : \mathbb{R} \to \mathcal{L}(\mathbb{R}^d)$ *be a locally integrable function with bounded growth and spectrum* $\Sigma(A) = [a_1, b_1] \cup \dots \cup [a_n, b_n]$ *such that* A *is in blockdiagonal form*

$$A(t) = \operatorname{diag}(A_1(t), \dots, A_n(t))$$

with

$$A_i : \mathbb{R} \to \mathcal{L}(\mathbb{R}^{d_i})$$

and $d_1 + \dots + d_n = d$ *and*

$$\Sigma(A_i) = [a_i, b_i], i = 1, \dots n.$$

Let $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ *be a function which fulfills the following properties:*

1. *The mapping* $f(t, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ *is continuous for almost all* $t \in \mathbb{R}$.
2. *The mapping* $f(\cdot, x) : \mathbb{R} \to \mathbb{R}^d$ *is measurable for all* $x \in \mathbb{R}^d$.
3. *For almost all* $t \in \mathbb{R}$ *and all* $x, y \in \mathbb{R}^d$ *we have*

$$F(t, 0) = 0$$
$$\|F(t, x) - F(t, y)\| \le L(x - y)$$

with a constant $L \ge 0$.

Then the ordinary differential equation

$$\dot{x} = A(t)x + F(t, x) \tag{6.5}$$

is called (reduced) standard system . *We denote the solution which starts at time* $\tau \in \mathbb{R}$ *in point* $x \in \mathbb{R}^d$ *by* $\mu(\cdot, \tau, x)$. *We assume, that for all* $t, \tau \in \mathbb{R}$ *the solution* $\mu(t, \tau, x)$ *exists.*

Notations

Now we will define useful abbreviations and notations for the standard system (6.5).

$$\mathbb{R}^d \supset E_i := \{0\} \times \ldots \times \{0\} \times \mathbb{R}^{d_i} \times \{0\} \times \ldots \times \{0\} \cong \mathbb{R}^{d_i}$$

and

$$E_i \ni (0, \ldots, 0, x_i, 0, \ldots, 0) \cong x_i \in \mathbb{R}^{d_i}$$

for $i = 1, \ldots n$. For i, j with $1 \leq i \leq j \leq n$ we define

$$E_{ij} := \{0\} \times \ldots \times \{0\} \times \mathbb{R}^{d_i} \times \ldots \times \mathbb{R}^{d_j} \times \{0\} \times \ldots \times \{0\} \cong \mathbb{R}^{d_i} \times \ldots \times \mathbb{R}^{d_j}$$

which is the direct product $E_i \oplus \ldots \oplus E_j$ and

$$E_{ij}^{\perp} := \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_{i-1}} \times \{0\} \times \ldots \times \{0\} \times \mathbb{R}^{d_{j+1}} \times \ldots \times \mathbb{R}^{d_n}$$

the orthogonal complement of this direct product. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_n}$ we define

$$x_{ij} := (x_i, \ldots x_j) \in \mathbb{R}^{d_i} \times \ldots \times \mathbb{R}^{d_j} \cong E_{ij}$$

and

$$x_{ij}^{\perp} := (x_1, \ldots, x_{i-1}, x_{j+1}, \ldots, x_n) \in \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_{i-1}} \times \mathbb{R}^{d_{j+1}} \times \ldots \times \mathbb{R}^{d_n} \cong E_{ij}^{\perp}.$$

We will use in this chapter the maximum norm in $\mathbb{R}^d$, i.e. we have

$$\|x_{ij}\| = \max_{i \leq k \leq j} \|x_k\| \text{ and } \left\|x_{ij}^{\perp}\right\| = \max_{k \notin \{i, \cdots, j\}} \|x_k\| \text{ with } x_k \in \mathbb{R}^{d_k}.$$

One can write the standard system (6.5) componentwise in $\mathbb{R}^d = E_1 \times \ldots \times E_n$ as

$$\begin{aligned}
\dot{x}_1 &= A_1(t)x_1 + F_1(t, x_1, \ldots, x_n) \\
\dot{x}_2 &= A_2(t)x_2 + F_2(t, x_1, \ldots, x_n) \\
&\vdots \\
\dot{x}_n &= A_n(t)x_n + F_n(t, x_1, \ldots, x_n)
\end{aligned}$$

with the component functions $F_i : \mathbb{R} \times \mathbb{R}^d \to E_i$ of $F = (F_1, \ldots, F_n)$. We can combine for every $j \in \{1, \ldots, n-1\}$ the first j equations and the remaining $n - j$ equations to get a standard system which has two parts

$$\begin{array}{rclcl}
\dot{x}_{1j} & = & A_{1j}(t)x_{1j} & + & F_{1j}(t, x) \\
\dot{x}_{j+1,n} & = & A_{j+1,n}(t)x_{j+1,n} & + & F_{j+1,n}(t, x).
\end{array}$$

This kind of system will be of great importance in the next section, where we do not need a finer splitting.

Choosing the Constants

For getting the global existence of integral manifolds, we first have to fix some constants. They all have to do with the spectral intervals $[a_1, b_1], \ldots, [a_n, b_n]$.

- For each spectral interval $[a_i, b_i], i = 1, \dots n$ we choose $\alpha_i, \beta_i \in \mathbb{R}$ with

$$\alpha_i < a_i \text{ and } b_i < \beta_i,$$

such that the intervals $[\alpha_1, \beta_1], \dots, [\alpha_n, \beta_n]$ are pairwise disjoint and in the hyperbolic case (if none of the spectral intervals $[a_i, b_i]$ contains 0) none of these intervals contain 0.

- We choose $\delta \in \mathbb{R}$ with

$$0 < \delta < \min\left\{\frac{\alpha_2 - \beta_1}{2}, \dots, \frac{\alpha_n - \beta_{n-1}}{2}\right\},$$

i.e. the intervals $[\alpha_i - \delta, \beta_i + \delta], i = 1, \dots, n$ are disjoint. In the hyperbolic case, none of them contains 0.

Finally we choose a $K \geq 1$ which depends on these constants, according to the following lemma.

Lemma 6.1.11 *For the standard system (6.5) with spectrum $\Sigma(A) = [a_1, b_1] \cup \dots \cup [a_n, b_n]$ there is a $K \geq 1$ which depends upon the chosen α_i and β_i such that we have*

$$\begin{aligned} \|\eta_i(t, \tau)\| &\leq K e^{\beta_i(t-\tau)} \quad \textit{for } t \geq \tau, \\ \|\eta_i(t, \tau)\| &\leq K e^{\alpha_i(t-\tau)} \quad \textit{for } t \leq \tau, \end{aligned}$$

for every $i \in \{1, \dots, n\}$. For every $j \in \{1, \dots, n-1\}$ we get with the following abbreviations

$$\eta_{1j} = \operatorname{diag}(\eta_1, \dots, \eta_j) \textit{ and } \eta_{j+1,n} = \operatorname{diag}(\eta_{j+1}, \dots, \eta_n)$$

the inequalities

$$\begin{aligned} \|\eta_{1j}(t, \tau)\| &\leq K e^{\beta_j(t-\tau)} \quad \textit{for } t \geq \tau, \\ \|\eta_{j+1,n}(t, \tau)\| &\leq K e^{\alpha_{j+1}(t-\tau)} \quad \textit{for } t \leq \tau. \end{aligned}$$

Proof. See Lemma 4.6 in Siegmund [29]. ∎

Now we come to the definition of invariant fibre bundles. These are subsets of the extended phase space $\mathbb{R} \times \mathbb{R}^d$.

Definition 6.1.12 *Consider an ordinary differential equation*

$$\dot{x} = f(t, x) \tag{6.6}$$

with a Carathéodory function $f : D \subset \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. Then a topological manifold $\mathcal{W} \subset D$ is called an invariant fibre bundle *or* integral manifold *of (6.6) if it consists of solution curves, i.e. if for all $\tau \in \mathbb{R}$ and all $x \in \mathbb{R}^d$ with $(\tau, x) \in \mathcal{W}$ the inclusion $(t, \mu(t, \tau, x)) \in \mathcal{W}$ holds for all t in the maximal existence interval of $\mu(\cdot, \tau, x)$. For $\tau \in \mathbb{R}$ the set $\mathcal{W}(\tau) := \{x \in \mathbb{R}^d : (\tau, x) \in \mathcal{W}\}$ is called* fibre *of $\mathcal{W}$.*

After having made these definitions, we can finally state the existence theorem for the integral manifolds of the standard system.

Theorem 6.1.13 *Consider the standard system*

$$\dot{x} = A(t)x + F(t,x) \tag{6.7}$$

and suppose, that the inequality

$$KL < \delta$$

holds.

Then there exist two hierarchies

$$\begin{array}{lll} \mathbb{R}\times\{0\} \subset \mathcal{W}_1 \subset & \mathcal{W}_{1,2} & \subset \ldots \subset \mathcal{W}_{1,n} = \mathbb{R}\times\mathbb{R}^d \\ \mathbb{R}\times\{0\} \subset \mathcal{W}_n \subset & \mathcal{W}_{n-1,n} & \subset \ldots \subset \mathcal{W}_{1,n} = \mathbb{R}\times\mathbb{R}^d \end{array}$$

of invariant fibre bundles of (6.7). For $j = 1, \ldots, n-1$ the invariant fibre bundles are graphs

$$\mathcal{W}_{1j} = \{(\tau,x) \in \mathbb{R}\times\mathbb{R}^d : x_{j+1,n} = x_{1j}^{\perp} = w_{1j}(\tau, x_{1j})\}$$
$$\mathcal{W}_{j+1,n} = \{(\tau,x) \in \mathbb{R}\times\mathbb{R}^d : x_{1j} = x_{j+1,n}^{\perp} = w_{j+1,n}(\tau, x_{j+1,n})\}$$

of continuous mappings

$$\begin{array}{lllll} w_{1j} & : & \mathbb{R}\times E_{1j} & \to & E_{j+1,n} = E_{1j}^{\perp} \\ w_{j+1,n} & : & \mathbb{R}\times E_{j+1,n} & \to & E_{1j} = E_{j+1,n}^{\perp}. \end{array}$$

Furthermore the following statements are fulfilled:

(a) The integral manifolds a characterized uniquely by

$$\mathcal{W}_{1j} = \{(\tau,x) : \mu(\cdot,\tau,x) \text{ is } \gamma^{+}\text{-quasibounded}\}$$
$$\mathcal{W}_{j+1,n} = \{(\tau,x) : \mu(\cdot,\tau,x) \text{ is } \gamma^{-}\text{-quasibounded}\}$$

for every $\gamma \in (\beta_j + KL, \alpha_{j+1} - KL)$.

(b) The following invariance equalities hold for all $t \in \mathbb{R}$:

$$\begin{array}{lllll} \mu_{j+1,n}(t,\tau,x) & = & w_{1j}(t, \mu_{1j}(t,\tau,x)) & \text{for } (\tau,x) \in \mathcal{W}_{1j} \\ \mu_{1j}(t,\tau,x) & = & w_{j+1,n}(t, \mu_{j+1,n}(t,\tau,x)) & \text{for } (\tau,x) \in \mathcal{W}_{j+1,n}. \end{array}$$

(c) For all $\tau \in \mathbb{R}, x, y \in \mathbb{R}^d$ the following inequalities hold:

$$\|w_{1j}(\tau, x_{1j}) - w_{1j}(\tau, y_{1j})\| \le \frac{K^2L}{\delta - KL}\|x_{1j} - y_{1j}\|,$$

$$\|w_{j+1,n}(\tau, x_{j+1,n}) - w_{j+1,n}(\tau, y_{j+1,n})\| \le \frac{K^2L}{\delta - KL}\|x_{j+1,n} - y_{j+1,n}\|.$$

Furthermore we have $w_{1j}(\tau,0) = 0$ and $w_{j+1,n}(\tau,0) = 0$.

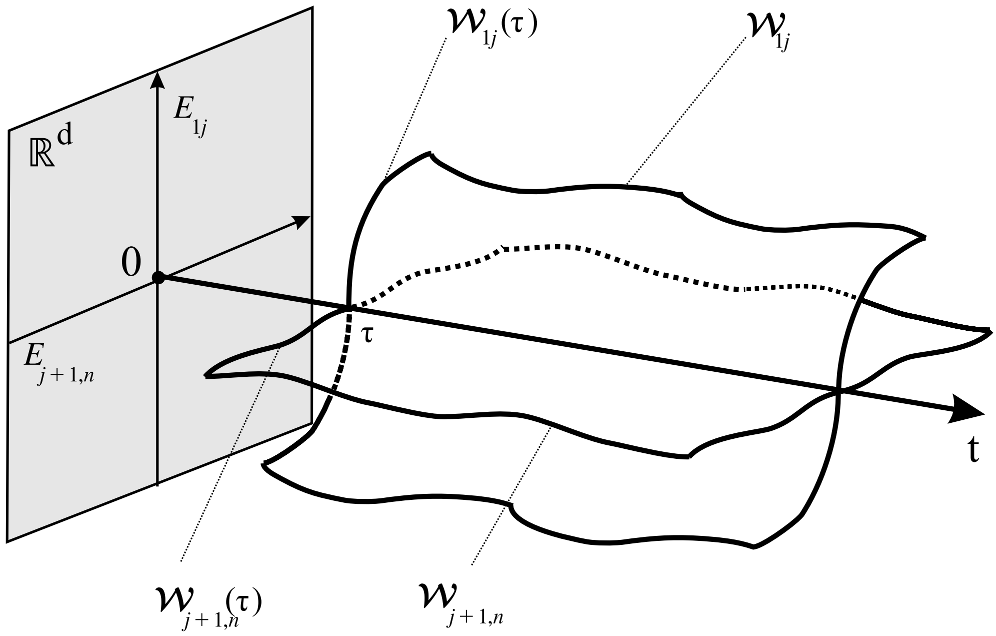

Figure 6.1: The integral manifolds manifolds $\mathcal{W}_{1j}$ and $\mathcal{W}_{j+1,n}$.

(d) *If the right hand side of (6.7) is periodic in t with period Θ, then the mappings w_{1j} and $w_{j+1,n}$ are periodic with period Θ, too.*

(e) *Only the trivial solution of (6.7) is contained on $\mathcal{W}_{1j}$ and $\mathcal{W}_{j+1,n}$, i.e. we have*

$$\mathcal{W}_{1j} \cap \mathcal{W}_{j+1,n} = \{(\tau, 0) : \tau \in \mathbb{R}\}.$$

In other words: The trivial solution is the only one, which is $\gamma^{\pm}$-quasibounded for all $\gamma \in (\beta_j + KL, \alpha_{j+1} - KL)$.

Proof. See Satz 4.16 in Siegmund [29]. ■

Remark 6.1.14 *Note, that the definition of the invariant fibre bundles $\mathcal{W}_{1j}$ and $\mathcal{W}_{j+,n}, n = 1, \ldots\ n-1$ is independent of the choice of the α_i, β_i, K and δ, i.e. the integral manifolds are* unique *for the system (6.7). For showing this for $k = 1, 2$ choose*

$$\alpha_i^k < a_i \text{ and } \beta_i^k > b_i \text{ for } i = 1, \ldots, n,$$

and δ^k

$$0 < \delta^k < \min\left\{\frac{\alpha_2^k - \beta_1^k}{2}, \ldots, \frac{\alpha_n^k - \beta_{n-1}^k}{2}\right\}$$

and choose K^k according to Lemma 6.1.11. Furthermore suppose, that $K^k L^k < \delta^k$, where L^k are two Lipschitz constants according to Definition 6.1.10. Then there exist the invariant fibre bundles

$$\mathcal{W}^1_{1j} = \{(\tau, x) : \mu(\cdot, \tau, x) \text{ is } \gamma^+\text{-quasibounded}\}$$
$$\mathcal{W}^1_{j+1,n} = \{(\tau, x) : \mu(\cdot, \tau, x) \text{ is } \gamma^-\text{-quasibounded}\}$$

for all $\gamma \in (\beta^1_j + K^1 L^1, \alpha^1_{j+1} - K^1 L^1) =: \Gamma^1_j$ and

$$\mathcal{W}^2_{1j} = \{(\tau, x) : \mu(\cdot, \tau, x) \text{ is } \gamma^+\text{-quasibounded}\}$$
$$\mathcal{W}^2_{j+1,n} = \{(\tau, x) : \mu(\cdot, \tau, x) \text{ is } \gamma^-\text{-quasibounded}\}$$

for all $\gamma \in (\beta^2_j + K^2 L^2, \alpha^2_{j+1} - K^2 L^2) =: \Gamma^2_j$. Now suppose, that for an $j \in \{1, \dots, n-1\}$ we have $\beta^1_j + K^1 L^1 < \beta^2_j + K^2 L^2$. For every $(\tau, x) \in \mathcal{W}_{1j}$ we know, that $\mu(\cdot, \tau, x)$ is γ^+-quasibounded for every $\gamma \in \Gamma^1$. But then $\mu(\cdot, \tau, x)$ is γ^+-quasibounded for $\gamma \in \Gamma^2$, because then $\gamma > \beta^1_j + K^1 L^1$. Therefore $(\tau, x) \in \mathcal{W}^2_{1j}$ and it follows, that

$$\mathcal{W}^1_{1j} \subseteq \mathcal{W}^2_{1j}. \tag{6.8}$$

Every $(\tau, x) \in \mathcal{W}^1_{1j}$ can be written as

$$(\tau, x) = (\tau, x_{1j}, w^1_{1j}(\tau, x_{1j}))$$

and every $(\tau, y) \in \mathcal{W}^2_{1j}$ as

$$(\tau, y) = (\tau, y_{1j}, w^2_{1j}(\tau, y_{1j})).$$

For $(\tau, y) \in \mathcal{W}^2_{1j}$ we get by (6.8) that $(\tau, y_{1j}, w^1_{1j}(\tau, y_{1j})) \in \mathcal{W}^1_{1j} \subseteq \mathcal{W}^2_{1j}$. Thus

$$(\tau, y_{1j}, w^1_{1j}(\tau, y_{1j})) = (\tau, y_{1j}, w^2_{1j}(\tau, y_{1j}))$$

and it follows, that $\mathcal{W}^1_{1j} = \mathcal{W}^2_{1j}$. The argumentation for $\mathcal{W}_{j+1,n}$ is similar.

The following Lemma shows, that we can even get a quasiboundedness result for every solution of the standard system.

Corollary 6.1.15 *Under the assumptions of Theorem 6.1.13 it follows, that for every $(\tau, x) \in \mathbb{R} \times \mathbb{R}^d$ the solution $\mu(\cdot, \tau, x)$ is γ^+-quasibounded for every $\gamma \in (\beta_n + KL, \infty)$, and γ^--quasibounded for every $\gamma \in (-\infty, \alpha_1 - KL)$. In particular, the standard system (6.7) is asymptotically stable, if $\beta_n + KL < 0$.*

Proof. Define the *extended* system on $\mathbb{R}^d \times \mathbb{R}$ by

$$\begin{aligned} \dot{x} &= A(t) + F(t, x) \\ \dot{y} &= \lambda y \end{aligned} \tag{6.9}$$

where $\lambda \in (\beta_n + 3\delta, \infty)$. Denote the solutions of (6.9) by

$$\mu^{ext}(\cdot, \tau, x, y) = (\mu(\cdot, \tau, x), e^{\lambda(t-\tau)}y)$$

for $(x, y) \in \mathbb{R}^d \times \mathbb{R}$. The corresponding *extended linear* system on $\mathbb{R}^d \times \mathbb{R}$ is defined by

$$\begin{aligned} \dot{x} &= A(t)x \\ \dot{y} &= \lambda y \end{aligned} \tag{6.10}$$

where we denote the fundamental solution of (6.10) by

$$\eta^{ext}(t, \tau) = \operatorname{diag}(\eta(t, \tau), e^{\lambda(t-\tau)}).$$

Note, that the extended linear system fulfills the corresponding relations of Lemma 6.1.11 with $\alpha_{n+1} < \lambda$ and $\alpha_{n+1} - \delta > \beta_n + \sigma$ and the constant K of the standard system (6.5). Thus we can apply the existence Theorem 6.1.13 and get the unique integral manifold

$$\mathcal{W}_{1n}^{ext} := \{(\tau, x, y) \in \mathbb{R}^d \times \mathbb{R} : \mu^{ext}(\cdot, \tau, x, y) \text{ is } \gamma^+\text{-quasibounded}\}$$

for every $\gamma \in (\beta_n + KL, \alpha_{n+1} - KL)$. Because $\mu^{ext}(\cdot, \tau, x, y) = (\mu(\cdot, \tau, x), e^{\lambda(t-\tau)}y)$ it follows, that $y^{ext}(\tau, x, y)$ is γ^+-quasibounded if and only if $y = 0$ because $\gamma < \lambda$. Thus it follows, that $\mathcal{W}_{1n}^{ext} = \mathbb{R}^d \times \{0\}$, and the assertion follows.

The other characterization of the standard system follows by considering the extended system with $\lambda \in (-\infty, \alpha_1 - KL)$. ■

The linear integral manifolds $\mathcal{W}_{ij}, 1 \leq i \leq j \leq n$ of a *linear* (reduced) standard system $\dot{x} = A(t)x$ can be defined by $\mathcal{W}_{ij} = \mathcal{W}_{1j} \cap \mathcal{W}_{in}$. The following proposition generalizes this for the nonlinear standard system (6.5) to define the integral manifolds $\mathcal{W}_{ij}$.

Theorem 6.1.16 *Consider the (reduced) standard system*

$$\dot{x} = A(t)x + F(t, x) \tag{6.11}$$

and suppose, that the following inequality holds:

$$KL < \frac{\delta}{K+1}. \tag{6.12}$$

Then for every i, j with $1 \leq i \leq j \leq n$ the intersection

$$\mathcal{W}_{ij} := \mathcal{W}_{1j} \cap \mathcal{W}_{in}$$

is an invariant fibre bundle of (6.11). For every $(i, j) \neq (1, n)$ the integral manifold $\mathcal{W}_{ij}$ is a graph

$$\mathcal{W}_{ij} := \{(\tau, x) \in \mathbb{R} \times \mathbb{R}^d : x_{ij}^{\perp} = w_{ij}(\tau, x_{ij})\}$$

of a continuous mapping

$$w_{ij} : \mathbb{R} \times E_{ij} \to E_{ij}^{\perp}$$

and the following statements are satisfied:

(a) *If $i > 1$ and $j < n$ then the integral manifold $\mathcal{W}_{ij}$ is uniquely characterized by*

$$\mathcal{W}_{ij} = \{(\tau, x) : \mu(\cdot, \tau, x) \text{ is } \gamma_i^- \text{- and } \gamma_j^+ \text{-quasibounded}\}$$

for every $\gamma_i \in (\beta_{i-1} + KL, \alpha_i - KL)$ and $\gamma_j \in (\beta_j + KL, \alpha_{j+1} - KL)$.

(b) *The invariance equation*

$$\lambda_{ij}^{\perp}(\tau, x) = w_{ij}(t, \mu_{ij}(t, \tau, x))$$

is satisfied for every $t \in \mathbb{R}$ and $(\tau, x) \in \mathcal{W}_{ij}$.

(c) *For every $\tau \in \mathbb{R}, x_{ij}, y_{ij} \in E_{ij}$ the inequality*

$$\|w_{ij}(\tau, x_{ij}) - w_{ij}(\tau, y_{ij})\| \leq \frac{K^2 L}{\delta - KL} \|x_{ij} - y_{ij}\|$$

is satisfied and we have $w_{ij}(\tau, 0) = 0$.

(d) *If the right hand side of (6.11) is periodic in t with period Θ, then the mapping w_{ij} is also periodic in t with the same period.*

Proof. Cf. Satz 4.19 in Siegmund [29]. ■

Remark 6.1.17 *Note, that the integral manifolds are* uniquely *defined for the system (6.11). Because the $\mathcal{W}_{1j}$ and $\mathcal{W}_{j+1,n}$ are uniquely defined according to Remark 6.1.14, their intersections are unique, too.*

Up to now, we only considered invariant fibre bundles through the trivial solution in 0. We will now make the generalization of integral manifolds through arbitrary solutions.

Theorem 6.1.18 *Consider the (reduced) standard system*

$$\dot{x} = A(t)x + F(t, x) \tag{6.13}$$

and suppose, that the following inequality holds:

$$KL < \frac{\delta}{K + 1}. \tag{6.14}$$

Then there exists for every i, j with $1 \leq i \leq j \leq n, (i, j) \neq (1, n)$ an unique mapping $w_{ij} : \mathbb{R} \times E_{ij} \times \mathbb{R} \times \mathbb{R}^d \to E_{ij}^{\perp}$, such that for every $\tau^* \in \mathbb{R}$ and every $x^* \in \mathbb{R}^d$ the graph

$$\mathcal{W}_{ij}[\tau^*, x^*] = \{(\tau, x) \in \mathbb{R} \times \mathbb{R}^d : x_{ij}^{\perp} = w_{ij}(\tau, x_{ij}, [\tau^*, x^*])\}$$

of the mapping $w_{ij}(\cdot, \cdot, [\tau^*, x^*])$ can be characterized in the following way:
For $1 < i \leq j < n$ we have

$$\mathcal{W}_{ij}[\tau^*, x^*] = \{(\tau, x) : \mu(\cdot, \tau, x) - \mu(\cdot, \tau^*, x^*) \text{ is } \gamma_i^- \text{- and } \gamma_j^+ \text{-quasibounded}\}$$

for every $\gamma_i \in (\beta_{i-1} + KL, \alpha_i - KL)$ *and* $\gamma_j \in (\beta_j + KL, \alpha_{j+1} - KL)$.

For $i = 1$ *we have*

$$\mathcal{W}_{1j}[\tau^*, x^*] = \{(\tau, x) : \mu(\cdot, \tau, x) - \mu(\cdot, \tau^*, x^*) \text{ is } \gamma^+\text{-quasibounded}\}$$

for every $\gamma \in (\beta_j + KL, \alpha_{j+1} - KL)$.

For $j = n$ *we have*

$$\mathcal{W}_{in}[\tau^*, x^*] = \{(\tau, x) : \mu(\cdot, \tau, x) - \mu(\cdot, \tau^*, x^*) \text{ is } \gamma^-\text{-quasibounded}\}$$

for every $\gamma \in (\beta_{i-1} + KL, \alpha_i - KL)$. *The mapping* w_{ij} *has the following properties:*

(a) *For every* $\tau \in \mathbb{R}, x_{ij}, y_{ij} \in E_{ij}$ *and* $(\tau^*, x^*) \in \mathbb{R} \times \mathbb{R}^d$ *we have*

$$\|w_{ij}(\tau, x_{ij}, [\tau^*, x^*]) - w_{ij}(\tau, y_{ij}, [\tau^*, x^*])\| \leq \frac{K^2 L}{\delta - KL} \|x_{ij} - y_{ij}\| .$$

(b) *The graph* $\mathcal{W}_{ij}[\tau^*, x^*]$ *is an* invarian fibre bundles *or* integral manifold *of (6.13), the so called* $\mathcal{W}_{ij}$-integral manifold through (τ^*, x^*) or the solution $\varphi(\cdot, \tau^*, x^*)$.

(c) *The mapping* $w_{ij} : \mathbb{R} \times E_{ij} \times \mathbb{R} \times \mathbb{R}^d \to E_{ij}^{\perp}$ *is continuous.*

Proof. Satz 4.25 in [29]. ■

The following corollary is important for the results about the asymptotic phase (cf. Corollary 6.1.21). It supplies us with a sharper characterization of the quasiboundedness properties of trajectories in the integral manifold.

Corollary 6.1.19 *Let the conditions of Theorem 6.1.18 be fulfilled, and let* $j \in \{1, \ldots, n-1\}$. *Then for every* $\gamma \in (\beta_j + KL, \alpha_{j+1} - KL)$ *and every* $\tau \in \mathbb{R}, x \in \mathbb{R}^d$ *the following statements hold*

(a) *For* $(\tau, y) \in \mathcal{W}_{1j}[\tau, x]$ *we have*

$$\|\mu(t, \tau, x) - \mu(t, \tau, y)\| \leq \frac{K(\gamma - \beta_j)}{\gamma - \beta_j - KL} \|x_{1j} - y_{1j}\| e^{\gamma(t-\tau)} \text{ for } t \geq \tau,$$

$$\|\mu(t, \tau, x) - \mu(t, \tau, y)\| \geq \frac{\gamma - \beta_j - KL}{K(\gamma - \beta_j)} \|x_{1j} - y_{1j}\| e^{\gamma(t-\tau)} \text{ for } t \leq \tau.$$

(b) *For* $(\tau, y) \in \mathcal{W}_{j+1,n}[\tau, x]$ *we have*

$$\|\mu(t, \tau, x) - \mu(t, \tau, y)\| \leq \frac{K(\alpha_{j+1} - \gamma)}{\alpha_{j+1} - \gamma + KL} \|x_{j+1,n} - y_{j+1,n}\| e^{\gamma(t-\tau)} \text{ for } t \leq \tau,$$

$$\|\mu(t, \tau, x) - \mu(t, \tau, y)\| \geq \frac{\alpha_{j+1} - \gamma + KL}{K(\alpha_{j+1} - \gamma)} \|x_{j+1,n} - y_{j+1,n}\| e^{\gamma(t-\tau)} \text{ for } t \geq \tau.$$

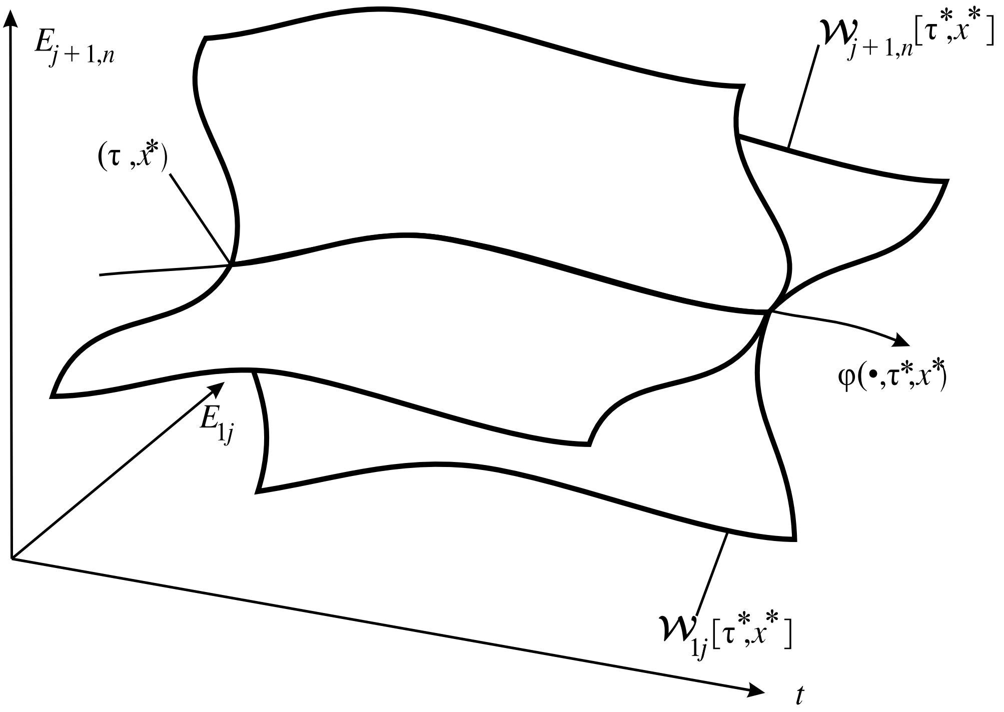

Figure 6.2: The integral manifolds $\mathcal{W}_{1j}[\tau^*, x^*]$ and $\mathcal{W}_{j+1,n}[\tau^*, x^*]$ through the solution $\varphi(\cdot, \tau^*, x^*)$.

Proof. We show the first inequality of (a), the other statements follow in the same way. Let $(\tau, y) \in \mathcal{W}_{1j}[\tau, x]$ and define

$$\nu(t) := \mu(t, \tau, y) - \mu(t, \tau, x).$$

Then $\nu(\cdot)$ is a solution of the ordinary differential equation

$$\dot{z} = A(t)z + \tilde{F}(t, z, [\tau, x]) \tag{6.15}$$

with

$$\tilde{F}(t, z, [\tau, x]) := F(t, z + \mu(t, \tau, x)) - F(t, \mu(t, \tau, x)).$$

This differential equation fulfills all the conditions of Theorem 6.1.13. Now we have to pick out some details of the proof of Theorem 6.1.13, which was made by Siegmund [29], Satz 4.16, page 116-128. First with $P = \mathbb{R} \times \mathbb{R}^d$, we define the fixed point operator T corresponding to the system (6.15) as in equation (4.22) on page 138 in [29], i.e.

$$T(\phi, z_{1j}, [\tau, x]) := \begin{pmatrix} \eta_{1j}(t, \tau) z_{1j} + \int_\tau^T \eta_{1j}(t, s) \tilde{F}_{1j}(s, \phi(s), [\tau, x]) ds \\ - \int_{-t}^{\infty} \eta_{j+1,n}(t, s) \tilde{F}_{j+1,n}(s, \phi(s), [\tau, x]) ds \end{pmatrix}$$

for $t \geq \tau$ and $\phi : [\tau, \infty) \to \mathbb{R}^d$ γ^+-quasibounded with $\phi_{1k}(\tau) = x_{1j}$. Let $\hat{\phi}(\cdot, \tau, y_{1j} - x_{1j}, [\tau, x])$ be the fixed point of $T(\cdot, y_{1j} - x_{1j}, [\tau, x])$. Then $\hat{\phi}$ is a solution of (6.15) by construction (see page 118 and Lemma 4.11 (a) in [29]). Furthermore we have

$$\hat{\phi}(\tau, \tau, y_{1j} - x_{1j}, [\tau, x]) = (y_{1j} - x_{1j}, \tilde{w}_{1j}(\tau, y_{1j} - x_{1j}, [\tau, x]))$$

where $\tilde{w}_{1j}$ is the function which defines the integral manifold $\tilde{\mathcal{W}}_{1j}$ of the system (6.15).

But as in the proof of Satz 4.25 in [29] on page 138 we get

$$w_{1j}(\tau, y_{1j}, [\tau, x]) = \tilde{w}_{1j}(\tau, y_{1j} - \mu_{1j}(\tau, \tau, x), [\tau, x]) + \mu_{1j}^{\perp}(\tau, \tau, x).$$

Here the w_{1j} is the one from Theorem 6.1.18. Thus it follows, that

$$\hat{\phi}(\tau, \tau, y_{1j} - x_{1j}, [\tau, x]) = \nu(\tau)$$

and the solutions $\hat{\phi}(\cdot, \tau, y_{1j} - x_{1j}, [\tau, x])$ and $\nu(\cdot)$ of (6.15) are identical. Now by relation (4.28) at page 122 in [29] we finally get

$$\begin{aligned} \|\mu(t, \tau, y) - \mu(t, \tau, x)\| &= \left\| \hat{\phi}(t, \tau, y_{1j} - x_{1j}, [\tau, x]) - \hat{\phi}(t, \tau, 0, [\tau, x]) \right\| \\ &\leq \frac{K(\gamma - \beta_j)}{\gamma - \beta_j - KL} \|y_{1j} - x_{1j}\| \, e^{\gamma_j (t-\tau)} \end{aligned}$$

for all $t \geq \tau$. ■

The preceding lemma supplies us with the remarkable result, that the differences $\mu(t, \tau, y) - \mu(t, \tau, x)$ are not only γ^+-quasibounded, we have also a unique constant in front of the $e^{\gamma(t-\tau)}$-term for *all* y lying on the invariant fibre bundle.

6.1.3 Asymptotic Phases

The concept of the *asymptotic phases* goes back to A.M.Lyapunov [23], where he developed this concept for periodic solutions of analytic differential equations. For invariant manifolds with asymptotic phases and further references see B.Aulbach [4]. Here we cite a theorem which can be found in Siegmund [29] for asymptotic phases of the hierarchy manifolds in Theorem 6.1.13.

Theorem 6.1.20 *Let the conditions of Theorem 6.1.18 be fulfilled. Then there exist for every* $j \in \{1, \dots, n-1\}$ *unique mappings*

$$\hat{\mathcal{P}}_{1j} : \mathbb{R} \times \mathbb{R}^d \to \mathcal{W}_{1j} \text{ and } \hat{\mathcal{P}}_{j+1,n} : \mathbb{R} \times \mathbb{R}^d \to \mathcal{W}_{j+1,n},$$

the so called asymptotic phases *for* $\mathcal{W}_{1j}$ *and* $\mathcal{W}_{j+1,n}$ *with the following properties:*

(a) There are mappings $\mathcal{P}_{1j}, \mathcal{P}_{j+1,n} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ *with*

$$\hat{\mathcal{P}}_{1j}(\tau, x) = (\tau, \mathcal{P}_{1j}(\tau, x))$$
$$\hat{\mathcal{P}}_{j+1,n}(\tau, x) = (\tau, \mathcal{P}_{j+1,n}(\tau, x))$$

for all $\tau \in \mathbb{R}, x \in \mathbb{R}^d$, *i.e.* $\hat{\mathcal{P}}_{1j}$ *and* $\hat{\mathcal{P}}_{j+1,n}$ *are fibre mappings. They map for every* $\tau \in \mathbb{R}$ *the fibre* $\{\tau\} \times \mathbb{R}^d$ *of the invariant fibre bundle* $\mathbb{R} \times \mathbb{R}^d$ *onto the corresponding fibre* $\{\tau\} \times \mathcal{W}_{1j}(\tau)$ *resp.* $\{\tau\} \times \mathcal{W}_{j+1,n}(\tau)$.

(b) For every $\tau \in \mathbb{R}, x \in \mathbb{R}^d$ *and every* $\gamma \in (\beta_j + KL, a_{j+1} - KL)$ *we have*

$$\mu(\cdot, \tau, x) - \mu(\cdot, \hat{\mathcal{P}}_{1j}(\tau, x)) \quad \text{is } \gamma^-\text{-quasi bounded,}$$
$$\mu(\cdot, \tau, x) - \mu(\cdot, \hat{\mathcal{P}}_{j+1,n}(\tau, x)) \quad \text{is } \gamma^+\text{-quasi bounded.}$$

(c) The mappings $\mathcal{P}_{1j}$ *and* $\mathcal{P}_{j+1,n}$ *(and therefore* $\hat{\mathcal{P}}_{1j}$ *and* $\hat{\mathcal{P}}_{j+1,n}$*, too) are continuous and for all* $\tau \in \mathbb{R}, x \in \mathbb{R}^d$ *the inequalities*

$$\|\mathcal{P}_{1j}(\tau, x)\| \leq \frac{2}{1-C} \|x\|, \tag{6.16}$$

$$\|\mathcal{P}_{j+1,n}(\tau, x)\| \leq \frac{2}{1-C} \|x\| \tag{6.17}$$

are satisfied with $C = \frac{K^2 L}{\delta - KL}$. *Furthermore for all* $\tau \in \mathbb{R}, x \in \mathcal{W}_{j+1,n}(\tau), y \in \mathcal{W}_{1j}(\tau)$ *we have* $\mathcal{P}_{1j}(\tau, x) = 0$ *and* $\mathcal{P}_{j+1,n}(\tau, y) = 0$.

(d) The mappings $\hat{\mathcal{P}}_{1j}$ *and* $\hat{\mathcal{P}}_{j+1,n}$ *are nonlinear projections, i.e.*

$$\hat{\mathcal{P}}_{1j} \circ \hat{\mathcal{P}}_{1j} = \hat{\mathcal{P}}_{1j} \text{ and } \hat{\mathcal{P}}_{j+1,n} \circ \hat{\mathcal{P}}_{j+1,n} = \hat{\mathcal{P}}_{j+1,n}.$$

(e) For an arbitrary solution μ *of (6.13) the mappings* $\mathcal{P}_{1j}(\cdot, \mu(\cdot))$ *and* $\mathcal{P}_{j+1,n}(\cdot, \mu(\cdot))$ *are solutions of (6.13), the so called* asymptotic phases *of* μ*, which lie in the invariant fibre bundle* $\mathcal{W}_{1j}$ *and* $\mathcal{W}_{j+1,n}$.

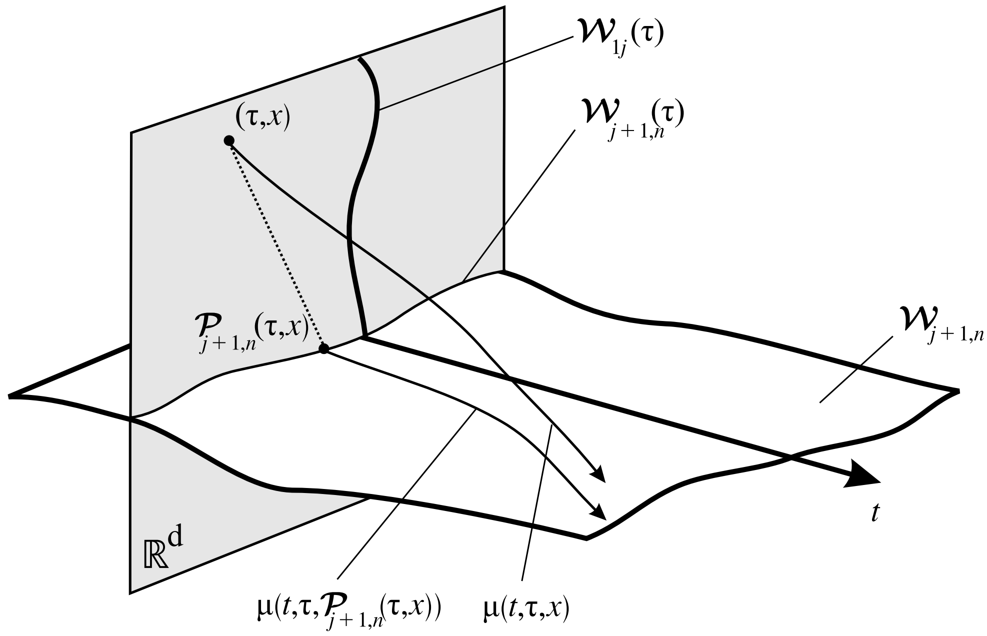

Figure 6.3: Asymptotic phase of (τ, x) in $\mathcal{W}_{j+1,n}$

(f) If the right hand side of (6.13) is periodic in t with period Θ, then so are the mappings $\mathcal{P}_{1j}$ and $\mathcal{P}_{j+1,n}$.

Proof. Satz 4.28 in Siegmund [29]. ∎

In the Figure 6.3 we have drawn a possible situation for an asymptotic phase in the case where $0 \in (\beta_j + KL, \alpha_{j+1} - KL)$. Then according to the preceding Theorem $\mu(\cdot,\tau,x) - \mu(\cdot,\tau,\mathcal{P}_{j+1,n}(\tau,x))$ is γ^+-quasibounded for a $\gamma < 0$. Thus the two trajectories converge together as t goes to infinity, but both trajectories are unbounded.

The following corollary characterizes the statement (b) in the preceding Theorem more precisely.

Corollary 6.1.21 *Let the conditions of Theorem 6.1.20 be fulfilled and let $j \in \{1, \dots, n-1\}$. Then for every $\gamma \in (\beta_j + KL, \alpha_{j+1} - KL)$ and every $\tau \in \mathbb{R}, x \in \mathbb{R}^d$ we have*

$$\|\mu(t,\tau,x) - \mu(t,\tau,\mathcal{P}_{1j}(\tau,x))\| \le \frac{K(\alpha_{j+1} - \gamma)}{\alpha_{j+1} - \gamma + KL} \|x - \mathcal{P}_{1j}(\tau,x)\| \, e^{\gamma(t-\tau)} \text{ for } t \le \tau, \tag{6.18}$$

$$\|\mu(t,\tau,x) - \mu(t,\tau,\mathcal{P}_{j+1,n}(\tau,x))\| \le \frac{K(\gamma - \beta_j)}{\gamma - \beta_j - KL} \|x - \mathcal{P}_{j+1,n}(\tau,x)\| \, e^{\gamma(t-\tau)} \text{ for } t \ge \tau, \tag{6.19}$$

Proof. Let $j \in \{1, \ldots, n-1\}$. According to Lemma 4.26 in [29] there is a unique mapping $g : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ with

$$\mathcal{W}_{1j}[\tau, x](\tau) \cap \mathcal{W}_{j+1,n}[\tau, y](\tau) = \{g(\tau, x, y)\}$$

for all $\tau \in \mathbb{R}, x, y \in \mathbb{R}^d$. For $y = 0$ we have

$$\mathcal{W}_{1j}(\tau) \cap \mathcal{W}_{j+1,n}[\tau, x](\tau) = \{g(\tau, 0, x)\}.$$

Then the mappings $\mathcal{P}_{1j}$ and $\hat{\mathcal{P}}_{1j}$ are defined by

$$\mathcal{P}_{1j}(\tau, x) := g(\tau, 0, x) \text{ and } \hat{\mathcal{P}}_{1j}(\tau, x) := (\tau, \mathcal{P}_{1j}(\tau, x))$$

(see for example the proof of Satz 4.28 in [29]). Now because $\hat{\mathcal{P}}_{1j}(\tau, x) \in \mathcal{W}_{j+1,n}[\tau, x]$ we get with Corollary 6.1.19 (b) the relation (6.18). The inequality (6.19) follows in the same way. ■

As in Corollary 6.1.19 the result is, that the difference $\mu(\cdot, \tau, x) - \mu(\cdot, \tilde{P}_{1j}(\tau, x))$ is not only quasibounded, we also have a constant in front of the $e^{\gamma(t-\tau)}$-term, which is independent of the chosen initial values.

6.1.4 The Hartman-Grobman Theorem

For autonomous differential equations with singular point, the theorem of Hartman-Grobman (cf. P.Hartman [18], D.M.Grobman [14]) is well known. It says, that hyperbolic C^1-systems are locally topologically conjugate to their linearization at their singular points. Here we cite the generalization for our nonautonomous standard system. In Siegmund [29] one can also find a generalization for nonhyperbolic systems (Satz 5.10).

Theorem 6.1.22 *Consider the (reduced) standard system*

$$\dot{x} = A(t)x + F(t, x) \tag{6.20}$$

with $\sup_{t \in \mathbb{R}, x \in \mathbb{R}^d} \|F(t, x)\| < \infty, KL < \frac{\delta}{K+1}$ *and*

$$0 \notin \Sigma(A).$$

Then the system (6.5) is topologically equivalent (with respect to the trivial solution) to the linear system

$$\dot{x} = A(t)x \tag{6.21}$$

via a topologically equivalence $\mathcal{H} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ *which maps the invariant fibre bundles* $\mathcal{W}_i, i = 1, \ldots, n$ *of (6.20) onto the corresponding linear integral manifolds* $\mathbb{R} \times E_i$ *of (6.21). In particular we have*

(a) The mapping $\mathcal{H}(\tau, \cdot) : \mathbb{R}^d \to \mathbb{R}^d, x \to \mathcal{H}(\tau, x)$ *is a homeomorphism for every* $\tau \in \mathbb{R}$, *with inverse* $\mathcal{H}^{-1}(\tau, \cdot)$, *and the mapping* $\mathcal{H}^{-1} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *is continuous.*

(b) *$\mathcal{H}(\cdot, \mu(\cdot))$ is a solution of (6.21) for every solution $\mu(\cdot)$ of (6.20).*

(c) *$\mathcal{H}^{-1}(\cdot, \nu(\cdot))$ is a solution of (6.20) for every solution $\nu(\cdot)$ of (6.21).*

(d) *$\mathcal{H}(\tau, 0) = \mathcal{H}^{-1}(\tau, 0) = 0$ for all $\tau \in \mathbb{R}$.*

(e) *If the right hand side of (6.20) is periodic in t with period Θ, then so are $\mathcal{H}$ and $\mathcal{H}^{-1}$.*

Proof. Cf. [29] Satz 5.9. ■

6.2 Local Theory for Periodic Systems

In this section we consider a special class of periodic nonlinear differential equations on $\mathbb{R}^d$. Consider the system

$$\dot{x} = A(t) + F(t, x). \tag{6.22}$$

where $A : \mathbb{R} \to \mathcal{L}(\mathbb{R}^d)$ is locally integrable and Θ-periodic. Suppose that

$$F(t, x) := F_0(x) + \sum_{i=1}^{m} u_i(t) F_i(x)$$

where $F_i : \mathbb{R} \to \mathbb{R}$ is continuous differentiable in the x-component and

$$\begin{aligned} F_i(t, 0) &= 0 \text{ and} \\ \frac{\partial F_i}{\partial x}(t, 0) &= 0 \text{ for all } t \in \mathbb{R}, i = 1, \dots, m, \end{aligned}$$

and

$$t \to u(t) := (u_1(t), \dots, u_m(t))$$

is a locally integrable and bounded function which is Θ-periodic. Such systems naturally occur for control affine systems with singular points. We will denote the system (6.22) as the *original* system and the solutions by $\varphi(\cdot, \tau, x)$. Note that this system is in fact a standard system in the sense of Definition 6.1.10.

Now in general such a system does not fulfill the assumptions of Theorem 6.1.13. Thus we can not apply this Theorem, to get the existence of (global) invariant fibre bundles through the trivial solution.

For obtaining at least local results near the fixed point 0, we apply the following reduction process on the original system. This will play an important part in the Chapter 3 and 4. Here we introduce the necessary notations for this chapters.

6.2.1 The Linearized System and Floquet Theory

Associated to the original system (6.22) is the *linearized* system

$$\dot{x} = A(t)x \tag{6.23}$$

We denote by $\eta(t,\tau)$ the fundamental solution of (6.23). Because $A(\cdot)$ is Θ-periodic we can apply Floquet Theory to transform the nonautonomous system in an autonomous one, see for example Sansone and Conti [27]. There is a matrix $Q \in \mathcal{L}(\mathbb{C})$ with $\eta(\Theta,0) = e^{\Theta Q}$. If we define $R := Q + \bar{Q} \in \mathbb{R}^{d\times d}$ we have $\eta(2\Theta,0) = e^{\Theta R}$ and we get

$$\eta(t,0) = g(t)e^{\frac{1}{2}tR}$$

where

$$g(t) := \eta(t,0)e^{-\frac{1}{2}tR} \tag{6.24}$$

is 2Θ-periodic and differentiable for almost all $t \in \mathbb{R}$..

The linear transformation

$$x = g(t)z$$

transforms the nonautonomous linear system (6.23) into the autonomous system

$$\dot{z} = Rz. \tag{6.25}$$

If we denote the fundamental solutions of (6.25) by $\eta(t,\tau,R) := e^{(t-\tau)R}$, then we have

$$\eta(t,\tau)g(\tau) = g(t)\eta(t,\tau,R).$$

Thus it follows, that $g(t)$ is a kinematic similarity transformation, compare Definition 6.1.8. The eigenvalues $\sigma_1,\dots,\sigma_d$ of $\eta(\Theta,0)$ are called *characteristic roots* or *characteristic multipliers* of the system (6.23). The eigenvalues $\xi_1,\dots,\xi_d$ of R which are determined by (6.23) only modulo $2\pi i$ are called *chararcteristic exponents* . Because σ_i^2 are the eigenvalues of $\eta(2\Theta,0)$ we have $\sigma_i = e^{\frac{1}{2}\Theta\xi_i}$ for $i = 1,\dots,d$ by numbering them in the right way. Note, that $\lambda_i := \mathrm{Re}(\xi_i)$ are just the Lyapunov exponents of (6.25), and we have

$$\Sigma(R) = \{\lambda_1,\dots,\lambda_d\}.$$

Because the transformation $g(t)$ is a kinematic transformation it follows by Proposition 6.1.9 that

$$\Sigma(R) = \Sigma(A)$$

and $\lambda_1,\dots,\lambda_d$ are the Lyapunov exponents of the nonautonomous system (6.23).

For the further construction of the invariant fibre bundles, we impose the following condition, which shall be valid for the rest of this section.

Condition 6.2.1 *There is a $k \in \{1, \dots, d-1\}$ such that the Lyapunov exponents of the linearized system (6.23) have the property*

$$\lambda_1 \geq \dots \geq \lambda_k > \lambda_{k+1} \geq \dots \geq \lambda_d.$$

Let X denote the sum of the generalized eigenspaces of R of the eigenvalues $\xi_1, \dots \xi_k$ and Y the sum of the generalized eigenspaces of the eigenvalues $\xi_{k+1}, \dots, \xi_d$. Then every point $z \in \mathbb{R}^d = X \oplus Y$ can be written in the form $z = x + y$ with unique $x \in X, y \in Y$. Now $\mathbb{R}^d = X \oplus Y$ and $X \times Y$ are algebraically and topologically isomorphic by means of the continuous isomorphism

$$\begin{array}{rccc} P: & \mathbb{R}^d & \to & X \times Y \\ & z & \mapsto & (P_X z, P_Y z) \end{array}$$

with the linear projections $P_X : \mathbb{R}^d \to X$ and $P_Y : \mathbb{R}^d \to Y$ onto X and Y. We define the function $P_X^{-1} : X \times \{0\} \to \mathbb{R}^d$ by $P_X^{-1}(x, 0) = x$ and $P_Y^{-1} : \{0\} \times Y \to \mathbb{R}^d$ by $P_Y^{-1}(0, y) = y$. Then $P^{-1}(x, y)^T = P_X^{-1}x + P_Y^{-1}$.

Remark 6.2.2 *If we use the norm $\|\cdot\|$ on $\mathbb{R}^d$ we define the* norm $\|\cdot\|_{X \times Y}$ *on $X \times Y$ by*

$$\|(x, y)\|_{X \times Y} := \max \left\{ \left\| P_X^{-1}(x, 0) \right\|, \left\| P_Y^{-1}(0, y) \right\| \right\}.$$

If it is clear, in which space we are and which norm we use, we just write $\|\cdot\|$ for the corresponding norm.

With the help of this projection we define the following blockdiagonal linear system on $X \times Y$

$$\begin{aligned} \dot{x} &= A^+ x \\ \dot{y} &= A^- y \end{aligned} \tag{6.26}$$

with $A^+ := P_X R P_X^{-1}$ and $A^- := P_Y R P_Y^{-1}$. The fundamental matrix of equation (6.26) will be denoted by

$$\nu(t, \tau) = \operatorname{diag}(\nu_X(t, \tau), \nu_Y(t, \tau))$$

where ν_X is the evolution operator of $\dot{x} = A^+ x$ and ν_Y of $\dot{y} = A^- y$. Note, that $\nu(t, \tau)(x, y)^T = P \circ \eta(t, \tau) \circ P^{-1}(x + y)$. It is clear, that $\Sigma(\operatorname{diag}(A^+, A^-)) = \Sigma(A)$.

Furthermore we define for the nonautonomous system (6.23) the subspaces

$$\begin{aligned} X(t) &:= g(t) X, \\ Y(t) &:= g(t) Y. \end{aligned} \tag{6.27}$$

for every $t \in \mathbb{R}$. Note, that we have $\mathbb{R}^d = X(t) \oplus Y(t)$ for all $t \in \mathbb{R}$.

6.2.2 The Transformed System

Define the transformation

$$\begin{array}{rccc}\mathcal{F}: & \mathbb{R}\times\mathbb{R}^d & \to & X\times Y \\ & (t,x) & \mapsto & P\circ g^{-1}(t)x,\end{array} \tag{6.28}$$

where for abbreviation we write

$$\mathcal{F}^{-1}(t) := (\mathcal{F}(t))^{-1} = (P\circ g^{-1}(t))^{-1} = g(t)\circ P^{-1} \text{ for all } t\in\mathbb{R}.$$

With the transformation $\mathcal{F}$, the original system (6.22) gets transformed into the following differential equation on $X\times Y$

$$\begin{aligned}\dot{x} &= A^+x + F^+(t,x,y)\\ \dot{y} &= A^-y + F^-(t,x,y)\end{aligned} \tag{6.29}$$

which we call the *transformed system* . Here $F^+ : \mathbb{R}\times X\times Y \to X$ and $F^- : \mathbb{R}\times X\times Y \to Y$ are Θ-periodic in the t-component and continuous differentiable functions in the (x,y)-component, defined by

$$F^+(t,x,y) := F_0^+(t,x,y) + \sum_{i=1}^m u_i(t)F_i^+(t,x,y), \tag{6.30}$$

$$F^-(t,x,y) := F_0^-(t,x,y) + \sum_{i=1}^m u_i(t)F_i^-(t,x,y), \tag{6.31}$$

where

$$\begin{aligned}F_i^+(t,x,y) &:= P_X\circ g^{-1}(t)\circ F_i(t,\mathcal{F}^{-1}(t,x,y)),\\ F_i^-(t,x,y) &:= P_Y\circ g^{-1}(t)\circ F_i(t,\mathcal{F}^{-1}(t,x,y)).\end{aligned}$$

Note, that the system (6.29) is also a standard system in the sense of Definition 6.1.10. The solutions starting at time τ at point $(x,y)\in X\times Y$ will be denoted $\psi(t,\tau,x,y)$ and they are unique and exist for all $t\in\mathbb{R}$.

Then we have the following relations between the transformed system (6.29) and the original system (6.22)

$$\begin{aligned}\varphi(t,\tau,z) &= \mathcal{F}^{-1}(t)\psi(t,\tau,\mathcal{F}(\tau)z)),\\ \psi(t,\tau,x,y) &= \mathcal{F}(t)\varphi(t,\tau,\mathcal{F}^{-1}(\tau)(x,y)^T)\end{aligned}$$

for every $(x,y)\in X\times Y, z\in\mathbb{R}^d$ and $t,\tau\in\mathbb{R}$.

Remark 6.2.3 *The norm $\|\cdot\|$ on $\mathbb{R}^d$ and the norm $\|\cdot\|_{X\times Y}$ on $X\times Y$ define the induced norms $\|\cdot\|_{\mathcal{L}(\mathbb{R}^d,X\times Y)}$ on $\mathcal{L}(\mathbb{R}^d,X\times Y)$ and $\|\cdot\|_{\mathcal{L}(X\times Y,\mathbb{R}^d)}$ on $\mathcal{L}(X\times Y,\mathbb{R}^d)$. For* *abbreviation we will write* $\|\mathcal{F}(t)\|$ *instead of* $\|\mathcal{F}(t)\|_{\mathcal{L}(\mathbb{R}^d, X\times Y)}$ *and* $\|\mathcal{F}^{-1}(t)\|$ *instead of* $\|\mathcal{F}^{-1}(t)\|_{\mathcal{L}(X\times Y,\mathbb{R}^d)}$ *and by definition we have*

$$\|\mathcal{F}(t)\| := \sup_{p\in\mathbb{R}^d, p\neq 0} \frac{\|\mathcal{F}(t)p\|_{X\times Y}}{\|p\|_{\mathbb{R}^d}}$$

and

$$\|\mathcal{F}^{-1}(t)\| := \sup_{(x,y)\in X\times Y, (x,y)\neq(0,0)} \frac{\|\mathcal{F}(t)(x,y)^T\|_{\mathbb{R}^d}}{\|(x,y)^T\|_{X\times Y}}.$$

Note that we then get for every $z \in \mathbb{R}^d, (x,y) \in X \times Y$ *and* $t \in \mathbb{R}$

$$\|\mathcal{F}(t)z\| \leq \|\mathcal{F}(t)\|\,\|z\| \;\; \textit{and} \;\; \|\mathcal{F}^{-1}(t)(x,y)^T\| \leq \|\mathcal{F}^{-1}(t)\|\,\|(x,y)^T\|$$

Now the function $t \to \|\mathcal{F}(t)\|$ *is continuous, and because it is periodic, it is even bounded. Hence we define*

$$\|\mathcal{F}\| := \sup_{t\in\mathbb{R}} \|\mathcal{F}(t)\| < \infty \;\; \textit{and} \;\; \|\mathcal{F}^{-1}\| := \sup_{t\in\mathbb{R}} \|\mathcal{F}^{-1}(t)\| < \infty.$$

6.2.3 The Reduced Standard System

Because we want to apply the existence Theorem 6.1.13 for invariant fibre bundles, we have to manipulate the transformed system (6.29) outside some neighborhood around the origin. We do this by applying a radial retraction to the nonlinear part of the equation.

For $\varepsilon > 0$ a *radial retraction* r_ε is a function $r_\varepsilon : X \times Y \to \operatorname{cl} B_\varepsilon(0)$, defined by

$$r_\varepsilon(x,y) := \begin{cases} (x,y) & \text{for } \|(x,y)\| \leq \varepsilon \\ \frac{\varepsilon}{\|(x,y)\|}(x,y) & \text{for } \|(x,y)\| > \varepsilon \end{cases}$$

The radial retraction r_ε is uniformly Lipschitz continuous with global Lipschitz constant 1 (see for example H.Amann [1]). If we define

$$F_\varepsilon^+(t,x,y) := F^+(t, r_\varepsilon(x,y)), \tag{6.32}$$
$$F_\varepsilon^-(t,x,y) := F^-(t, r_\varepsilon(x,y)), \tag{6.33}$$

for all $t \in \mathbb{R}$ and $(x,y) \in X \times Y$, we get the *reduced standard system*

$$\begin{aligned} \dot{x} &= Ax + F_\varepsilon^+(t,x,y) \\ \dot{y} &= By + F_\varepsilon^-(t,x,y) \end{aligned} \tag{6.34}$$

on $X \times Y$ which coincides on $\mathbb{R}\times \operatorname{cl} B_\varepsilon(0)$ with the transformed system (6.29). Note, that this differential equation is in fact a standard system in the sense of definition 6.1.10. The functions $F_\varepsilon^+ : \mathbb{R} \times X \times Y \to X, F_\varepsilon^- : \mathbb{R} \times X \times Y \to Y$ are Θ-periodic in the t-component and Lipschitz continuous in the x-component. The solutions starting at time τ at point $(x, y) \in X \times Y$ will be denoted $\mu_\varepsilon(t, \tau, x, y, u^h)$ and they are unique and exist for all $t \in \mathbb{R}$.

The radial restriction now supplies us with the necessary tool, to adjust the Lipschitz constants of the right hand side.

Lemma 6.2.4 *Let $L, Q > 0$. Then there exists an $\varepsilon := \varepsilon(L, Q) > 0$, such that the nonlinearities of the restricted system (6.34) fulfill the following properties:*

$$\begin{aligned}
\left\|F_\varepsilon^+(t,x,y) - F_\varepsilon^+(t,x',y')\right\| &\le L\left\|(x,y)-(x',y')\right\|, \\
\left\|F_\varepsilon^-(t,x,y) - F_\varepsilon^-(t,x',y')\right\| &\le L\left\|(x,y)-(x',y')\right\|, \\
\left\|F_\varepsilon^+(t,x,y)\right\| &\le Q, \\
\left\|F_\varepsilon^+(t,x,y)\right\| &\le Q,
\end{aligned}$$

for all $(x, y), (x', y') \in X \times Y$ and all $t \in \mathbb{R}$.

Proof. For each $t \in \mathbb{R}$ the function $(x, y) \mapsto F^+(t, x, y)$ is a differentiable function with continuous first derivative. Because of $\left.\frac{\partial F^+(t,x,y)}{\partial(x,y)}\right|_{(x,y)=(0,0)} = 0$ for each $t \in \mathbb{R}$ there exists an $\varepsilon(t) > 0$ with $\|F^+(t,x,y) - F^+(t,x',y')\| \le \frac{L}{2}\|(x,y)-(x',y')\|$ for all $(x, y), (x', y') \in \operatorname{cl} B_{\varepsilon(t)}(0), t \in \mathbb{R}$. Since the functions $F^+(t, x, y)$ are Θ-periodic in the t-variable and have the special structure (6.30), there exists an $\varepsilon > 0$ with $\varepsilon < \inf_{t\in[0,2\Theta]} \varepsilon(t)$. Thus we get

$$\left\|F^+(t,x,y) - F^+(t,x',y')\right\| \le L\left\|(x,y)-(x',y')\right\|$$

for all $(x, y), (x', y') \in \operatorname{cl} B_\varepsilon(0)$ and all $t \in \mathbb{R}$. This implies

$$\begin{aligned}
\left\|F_\varepsilon^+(t,x,y) - F_\varepsilon^+(t,x',y')\right\| &= \left\|F^+(t,r_\varepsilon(x,y)) - F^+(t,r_\varepsilon(x',y'))\right\| \\
&\le L\left\|r_\varepsilon(x,y) - r_\varepsilon(x',y'))\right\| \\
&\le L\left\|(x,y)^T - (x',y')^T\right\|
\end{aligned}$$

for all $(x, y), (x', y') \in X \times Y$ and all $t \in \mathbb{R}$. By choosing $\varepsilon > 0$ eventually smaller, we get by the same argument, that

$$\left\|F_\varepsilon^-(t,x,y) - F_\varepsilon^-(t,x',y')\right\| \le L\left\|(x,y)^T - (x',y')^T\right\|$$

By choosing ε small enough we can also achieve, that the mappings F_ε^+ and F_ε^- are bounded. ■

Remark 6.2.5 *The lemma shows, that for every given $L > 0$ there is an $\varepsilon(L) > 0$ such that the nonlinearities $F^+_{\varepsilon(L)}$ and $F^-_{\varepsilon(L)}$ of the restricted system (6.34) have L as Lipschitz constant. But the number $\varepsilon(L)$ is not uniquely defined, because for every $0 < \varepsilon < \varepsilon(L)$ the nonlinearities F_ε^+ and F_ε^- have also L as Lipschitz constant. To overcome this nonuniqueness we now introduce the following convention.*

Convention:
For every $L > 0$ we choose $\varepsilon(L) > 0$ such that the nonlinearities $F^+_{\varepsilon(L)}$ and $F^-_{\varepsilon(L)}$ have L as common Lipschitz constant and such that the function

$$\varepsilon : \mathbb{R} \to \mathbb{R}, L \mapsto \varepsilon(L)$$

is monotone decreasing. In particular we have $\lim_{L\to 0} \varepsilon(L) = 0$.

6.2.4 Local Invariant Fibre Bundles

In this last preparation step we choose constants which are related to the dichotomy spectrum and the Lipschitz constant, such that the assumptions of Theorem 6.1.13 are fulfilled.

- We choose $\alpha, \beta \in \mathbb{R}$ with

$$\lambda_{k+1} < \beta < \alpha < \lambda_k. \tag{6.35}$$

If $\lambda_{k+1} < 0 < \lambda_k$, then we choose α, β such that

$$\lambda_{k+1} < \beta < 0 < \alpha < \lambda_k. \tag{6.36}$$

and if $\lambda_1 < 0$, the we choose another $\hat{\beta} \in \mathbb{R}$ such that

$$\lambda_1 < \hat{\beta} < 0. \tag{6.37}$$

- We choose $\delta \in \mathbb{R}$ such that

$$0 < \delta < \frac{\beta - \alpha}{2}. \tag{6.38}$$

If $\lambda_{k+1} < 0 < \lambda_k$, then we choose $\delta \in \mathbb{R}$ eventually smaller such that

$$0 \in (\beta + \delta, \alpha - \delta).$$

If $\lambda_1 < 0$ then we choose $\delta \in \mathbb{R}$ eventually smaller such that

$$\hat{\beta} + \delta < 0.$$

- We choose $K = K(\alpha, \beta) \geq 1$ such that the following inequalities are satisfied

$$\begin{aligned} \|\nu_Y(t,\tau)\| &\leq K e^{\beta(t-\tau)} \quad \text{for } t \geq \tau, \\ \|\nu_X(t,\tau)\| &\leq K e^{\alpha(t-\tau)} \quad \text{for } t \leq \tau. \end{aligned} \tag{6.39}$$

This choice of K is possible according Lemma 6.1.11.

- For the chosen constants $\alpha, \beta, \hat{\beta}, K, \delta$ there is a $L^* > 0$, such that for every $0 < L \leq L^*$ the relation
$$KL < \frac{\delta}{K+1}. \tag{6.40}$$
is fulfilled. Note that we have
$$\left\| F^+_{\varepsilon(L)}(t,x,y) - F^+_{\varepsilon(L)}(t,x',y') \right\| \leq L \left\| (x,y) - (x',y') \right\|,$$
$$\left\| F^-_{\varepsilon(L)}(t,x,y) - F^-_{\varepsilon(L)}(t,x',y') \right\| \leq L \left\| (x,y) - (x',y') \right\|,$$
and we define $\varepsilon^* := \varepsilon(L^*)$.

Thus by choosing $L \in (0, L^*]$ we can apply Theorem 6.1.13 to the restricted system (6.34) with $\varepsilon := \varepsilon(L)$.

Definition 6.2.6 *Let $\varepsilon \in (0, \varepsilon^*]$. For the reduced standard system (6.34) we define*

(a) the unstable fibre bundle $\mathcal{X}_\varepsilon \subset \mathbb{R} \times X \times Y$ *which is characterized by*
$$\begin{aligned}\mathcal{X}_\varepsilon &= \{(\tau,x,y) \in \mathbb{R} \times X \times Y : \mu_\varepsilon(\cdot,\tau,x,y) \textit{ is } \gamma^-\textit{-quasibounded}\} \\ &= \{(\tau,x,y) \in \mathbb{R} \times X \times Y : y = w^+_\varepsilon(\tau,x)\}\end{aligned}$$
for every $\gamma \in (\beta + KL, \alpha - KL)$ with corresponding mapping $w^+_\varepsilon : \mathbb{R} \times X \to Y$ and

(b) the stable fibre bundle $\mathcal{Y}_\varepsilon \subset \mathbb{R} \times X \times Y$ *which is characterized by*
$$\begin{aligned}\mathcal{Y}_\varepsilon &= \{(\tau,x,y) \in \mathbb{R} \times X \times Y : \mu_\varepsilon(\cdot,\tau,x,y) \textit{ is } \gamma^+\textit{-quasibounded}\} \\ &= \{(\tau,x,y) \in \mathbb{R} \times X \times Y : x = w^-_\varepsilon(\tau,y)\}\end{aligned}$$
for every $\gamma \in (\beta + KL, \alpha - KL)$ with corresponding mapping $w^-_\varepsilon : \mathbb{R} \times Y \to X$.

Remark 6.2.7 *Note, that $\mathcal{X}_\varepsilon$ and $\mathcal{Y}_\varepsilon$ are uniquely defined and do not depend on the chosen α, β, K, δ by Remark 6.1.14.*

The maps $w^+_\varepsilon, w^-_\varepsilon$ are defined as in Theorem 6.1.13. We use here another notation than in Section 6.1. For our purposes a finer decomposition is not necessary.

Now with the help of the stable and unstable fibre bundles, which are global objects for the restricted system (6.34), we can define the local stable and unstable fibre bundles for the original system (6.22).

Definition 6.2.8 *Let $\varepsilon \in (0, \varepsilon^*]$. For the original system (6.22) we define*

(a) the local unstable fibre bundle $\mathcal{X}^{loc}_\varepsilon \subset \mathbb{R} \times \mathbb{R}^d$ *by*
$$\mathcal{X}^{loc}_\varepsilon := \{(\tau,x) \in \mathbb{R} \times \mathbb{R}^d : (\tau, \mathcal{F}(\tau)x) \in \mathcal{X}_\varepsilon\}$$

(b) and the local stable fibre bundle $\mathcal{Y}_\varepsilon^{loc} \subset \mathbb{R} \times \mathbb{R}^d$ *by*

$$\mathcal{Y}_\varepsilon^{loc} := \{(\tau, x) \in \mathbb{R} \times \mathbb{R}^d : (\tau, \mathcal{F}(\tau)x) \in \mathcal{Y}_\varepsilon\}.$$

The stable and unstable fibre bundles $\mathcal{X}_\varepsilon$ and $\mathcal{Y}_\varepsilon$ are invariant, i.e. for all $(\tau, x) \in \mathcal{X}_\varepsilon$ we have $\mu_\varepsilon(t, \tau, x) \in \mathcal{X}_\varepsilon(t)$ for every $t \in \mathbb{R}$. Now the local fibre bundles $\mathcal{X}_\varepsilon^{loc}, \mathcal{Y}_\varepsilon^{loc}$ are in general not invariant. But if we start in the hyperbolic case close enough to the singular point, then the corresponding trajectories stay in the fibre bundles.

Lemma 6.2.9 *Suppose, that $\lambda_{k+1} < 0 < \lambda_k$. Then there is an $\hat{\varepsilon} \in (0, \varepsilon^*]$ such that for every $\varepsilon \in (0, \hat{\varepsilon}]$ there is a neighborhood $W = W(\varepsilon) \subset \mathbb{R}^d$ such that for all $p \in \mathcal{X}_\varepsilon^{loc}(\tau) \cap W$ and all $q \in \mathcal{Y}_\varepsilon^{loc}(\tau) \cap W$ we have*

$$\varphi(t, \tau, p) = \mathcal{F}^{-1}(t)\psi(t, \tau, \mathcal{F}(\tau)p) = \mathcal{F}^{-1}(t)\mu_\varepsilon(t, \tau, \mathcal{F}(\tau)p) \text{ for all } t \leq \tau \text{ and}$$
$$\varphi(t, \tau, q) = \mathcal{F}^{-1}(t)\psi(t, \tau, \mathcal{F}(\tau)q) = \mathcal{F}^{-1}(t)\mu_\varepsilon(t, \tau, \mathcal{F}(\tau)q) \text{ for all } t \geq \tau.$$

In particular

$$\lim_{t \to -\infty} \varphi(t, \tau, p) = 0,$$
$$\lim_{t \to -\infty} \varphi(t, \tau, q) = 0.$$

Proof. Because $\lambda_{k+1} < 0$ we can choose $\alpha, \beta \in \mathbb{R}$ with (6.35) and $\beta < 0 < \alpha$. Choose $\delta, K \in \mathbb{R}$ such that (6.38) and (6.39) are fulfilled. Then there exists a $\hat{L} \in (0, L^*]$ such that (6.40) if fulfilled and

$$\beta + K\hat{L} < 0 < \alpha - K\hat{L}.$$

Define $\hat{\varepsilon} := \varepsilon(\hat{L})$. Now for every $(\tau, x, y) \in \mathcal{X}_\varepsilon$ we have by Corollary 6.1.19 for every $\gamma \in (\beta + K\hat{L}, \alpha - K\hat{L})$

$$\|\mu_\varepsilon(t, \tau, x, y)\| \leq \frac{K(\alpha - \gamma)}{\alpha - \gamma + K\hat{L}} \|(x, y)\| \, e^{\gamma(t-\tau)} \text{ for } t \leq \tau.$$

Thus by choosing $\gamma > 0$, we see that we can choose for every neighborhood $V' \subset X \times Y$ of 0 a neighborhood W' of 0 such that for every $(\tau, x, y) \in \mathcal{X}_{\hat{\varepsilon}} \cap W'$ we have

$$\mu_{\hat{\varepsilon}}(t, \tau, x, y) \in V' \text{ for all } t \leq \tau.$$

Again by Corollary 6.1.19 it follows, in the same way, that by choosing W' eventually smaller we have for every $(\tau, x, y) \in \mathcal{Y}_\varepsilon \cap W'$

$$\mu_{\hat{\varepsilon}}(t, \tau, x, y) \in V' \text{ for all } \tau \leq t.$$

Thus, for every $L \in (0, \hat{L}]$, we can find for every neighborhood $V' \subset X \times Y$ of 0 a neighborhood $W' \subset X \times Y$ of 0 such that for all $\tau \in \mathbb{R}, (x_1, y_1) \in \mathcal{X}_{\varepsilon(L)}(\tau) \cap W'$ and $(x_2, y_2) \in \mathcal{Y}_{\varepsilon(L)}(\tau) \cap W'$

$$\mu_{\varepsilon(L)}(t, \tau, x_1, y_1) \in V' \text{ for all } t \leq \tau \text{ and} \tag{6.41}$$
$$\mu_{\varepsilon(L)}(t, \tau, x_2, y_2) \in V' \text{ for all } \tau \leq t. \tag{6.42}$$

Now for $\varepsilon \in (0, \hat{\varepsilon}]$ choose a $L \in (0, \hat{L}]$ with $\varepsilon = \varepsilon(L)$ and choose the neighborhood $W' \subset X \times Y$ of 0 with (6.41) and (6.42) which belongs to $V' := B_\varepsilon(0) \subset X \times Y$. Because the restricted system (6.34) and the transformed system (6.29) coincide on $\operatorname{cl} B_\varepsilon(0)$, we get for all $\tau \in \mathbb{R}$ and $(x_1, y_1) \in \mathcal{X}_\varepsilon(\tau) \cap W'$ and $(x_2, y_2) \in \mathcal{Y}_\varepsilon(\tau) \cap W'$

$$\begin{aligned} \psi(t, \tau, x_1, y_1) &= \mu_\varepsilon(t, \tau, x_1, y_1) \text{ for all } t \leq \tau \text{ and} \\ \psi(t, \tau, x_2, y_2) &= \mu_\varepsilon(t, \tau, x_2, y_2) \text{ for all } \tau \leq t. \end{aligned}$$

Thus it follows, that for every $\tau \in \mathbb{R}$ and $p \in \mathcal{X}_\varepsilon^{loc}(\tau) \cap \mathcal{F}^{-1}(\tau) W'$ and $q \in \mathcal{Y}_\varepsilon(\tau) \cap \mathcal{F}^{-1}(\tau) W'$

$$\begin{aligned} \varphi(t, \tau, p) &= \mathcal{F}^{-1}(t) \psi(t, \tau, \mathcal{F}(\tau) p) \text{ for all } t \leq \tau \text{ and} \\ \varphi(t, \tau, q) &= \mathcal{F}^{-1}(t) \psi(t, \tau, \mathcal{F}(\tau) q) \text{ for all } \tau \leq t. \end{aligned}$$

Because $\mathcal{F}^{-1}$ is periodic it follows, that there is a neighborhood $W \subset \mathbb{R}^d$ of 0 such that $W \subset \mathcal{F}^{-1}(t) W'$ for all $t \in \mathbb{R}$. Then the assertion follows with this W. ■

If all the Lyapunov exponents are negative, then 0 is a locally asymptotic stable fixed point for the original system (6.22).

Lemma 6.2.10 *Suppose, that $\lambda_1 < 0$. Then there is an $\hat{\varepsilon} \in (0, \varepsilon^*]$ such that for every $\varepsilon \in (0, \hat{\varepsilon}]$ there is a neighborhood $W(\varepsilon) \subset \mathbb{R}^d$ of 0 such that for all $p \in W(\varepsilon)$ we have*

$$\varphi(t, \tau, p) = \mathcal{F}^{-1}(t) \psi(t, \tau, \mathcal{F}(\tau) p) = \mathcal{F}^{-1}(t) \mu_\varepsilon(t, \tau, \mathcal{F}(\tau) p) \text{ for all } t \geq \tau.$$

In particular for every neighborhood $V \subset \mathbb{R}^d$ of 0 there is a neighborhood $W \subset V$ of 0 such that for all $p \in W$

$$\begin{aligned} \varphi(t, \tau, p) &\in V \text{ for all } t \geq \tau, \\ \lim_{t \to \infty} \varphi(t, \tau, p) &= 0. \end{aligned}$$

Proof. Because $\lambda_d < 0$, we can choose $\alpha, \beta, \hat{\beta}$ with (6.35) and (6.37) such that

$$\hat{\beta} + \delta < 0.$$

Then there exists a $\hat{L} \in (0, L^*]$ such that (6.40) is fulfilled and $\hat{\beta} + KL < 0$. Define $\hat{\varepsilon} := \varepsilon(L)$. Then it follows by Corollary 6.1.15, that the restricted system is asymptotically stable for all $\varepsilon \in (0, \hat{\varepsilon}]$. We can find neighborhoods $V' \subset X \times Y$ and $W' \subset X \times Y$ of 0 such that for all $\tau \in \mathbb{R}$, $(x, y) \in W'$ we have

$$\begin{aligned} \mu_\varepsilon(t, \tau, x, y) &\in V' \text{ for all } t \geq \tau \text{ and} \\ \lim_{t \to \infty} \mu_\varepsilon(t, \tau, x, y) &= 0. \end{aligned}$$

By choosing $V' \subset B_\varepsilon(0)$, the assumptions follows as in the proof of Lemma 6.2.9. ■

If we fix $\varepsilon > 0$, then the (global) stable and unstable fibre bundles $\mathcal{X}_\varepsilon$ and $\mathcal{Y}_\varepsilon$ are unique for the reduced system (6.34) corresponding ε. But if we vary ε, we get different restricted systems, and it follows, that the fibre bundles for different ε also are different, i.e. if $\varepsilon \neq \varepsilon'$ then we have in general $\mathcal{X}_\varepsilon \neq \mathcal{X}_{\varepsilon'}$. Thus it follows, that the local stable and unstable fibre bundles also vary for different ε. The next lemma shows, that in the hyperbolic case at least locally around the fixed point 0 they coincide.

Proposition 6.2.11 *If $\lambda_{k+1} < 0 < \lambda_k$, then there exists a $\hat{\varepsilon} \in (0, \varepsilon^*]$, such that for every $\varepsilon \in (0, \hat{\varepsilon}]$ there exists a neighborhood $W \subset \mathbb{R}^d$ of 0 such that for all $\varepsilon' \in [\varepsilon, \hat{\varepsilon}]$ the following equations hold*

$$\mathcal{X}_\varepsilon^{loc}(t) \cap W = \mathcal{X}_{\varepsilon'}^{loc}(t) \cap W,$$
$$\mathcal{Y}_\varepsilon^{loc}(t) \cap W = \mathcal{Y}_{\varepsilon'}^{loc}(t) \cap W,$$

for all $t \in \mathbb{R}$.

Proof. Choose $\hat{\varepsilon}$ as in Lemma 6.2.9, and let $\varepsilon \in (0, \hat{\varepsilon}]$. Then as in the proof of the Lemma 6.2.9, there is a neighborhood $\tilde{W}(\varepsilon) \subset X \times Y$ of 0 such that for every $(\tau, x_1, y_1) \in \mathcal{X}_\varepsilon \cap \tilde{W}(\varepsilon)$ and $(\tau, x_2, y_2) \in \mathcal{Y}_\varepsilon \cap \tilde{W}(\varepsilon)$ we have

$$\mu_\varepsilon(t, \tau, x_1, y_1) \in B_\varepsilon(0) \text{ for all } t \leq \tau \text{ and}$$
$$\mu_\varepsilon(t, \tau, x_2, y_2) \in B_\varepsilon(0) \text{ for all } \tau \leq t.$$

Because for $\varepsilon' \in [\varepsilon, \hat{\varepsilon}]$ the restricted system which belongs to ε' coincides with the restricted system which belongs to ε on $\mathbb{R} \times B_\varepsilon(0)$ we have

$$\mu_\varepsilon(t, \tau, x_1, y_1) = \mu_{\varepsilon'}(t, \tau, x_1, y_1) \text{ for all } t \leq \tau \text{ and}$$
$$\mu_\varepsilon(t, \tau, x_2, y_2) = \mu_{\varepsilon'}(t, \tau, x_2, y_2) \text{ for all } t \leq \tau$$

for all $(\tau, x_1, y_1) \in \mathcal{X}_\varepsilon \cap \tilde{W}(\varepsilon)$ and $(\tau, x_2, y_2) \in \mathcal{Y}_\varepsilon \cap \tilde{W}(\varepsilon)$.

Choose $L, L' \in \mathbb{R}$ with $\varepsilon = \varepsilon(L)$ and $\varepsilon' = \varepsilon(L')$. Then because $L \leq L'$ it follows, that there is a $\gamma < 0$

$$\gamma \in (\beta + KL, \alpha - KL) \cap (\beta + KL', \alpha - KL').$$

For $(\tau, x_1, y_1) \in \tilde{W}(\varepsilon)$ the solution $\mu_{\varepsilon'}(t, \tau, x_1, y_1)$ is γ^--quasibounded if and only if $(x_1, y_1) \in \mathcal{X}_\varepsilon(\tau)$. Thus it follows

$$\mathcal{X}_\varepsilon(\tau) \cap \tilde{W}(\varepsilon) = \mathcal{X}_{\varepsilon'}(\tau) \cap \tilde{W}(\varepsilon).$$

On the other hand for $(\tau, x_2, y_2) \in \tilde{W}(\varepsilon)$ the solution $\mu_{\varepsilon'}(t, \tau, x_2, y_2)$ is γ^+-quasibounded if and only if $(x_2, y_2) \in \mathcal{Y}_{\varepsilon'}(\tau)$. Thus it follows

$$\mathcal{Y}_\varepsilon(t) \cap \tilde{W}(\varepsilon) = \mathcal{Y}_{\varepsilon'}(t) \cap \tilde{W}(\varepsilon).$$

As shown in the proof of Lemma 6.2.9 there is a neighborhood $W \subset \mathbb{R}^d$ of 0 such that $W \subset \mathcal{F}^{-1}(t)\tilde{W}(\varepsilon)$ for all $t \in \mathbb{R}$ and the assertion follows. ■

In the last part of this section, we show how close the (nonlinear) fibre bundles are to their linear analogon $X(t)$ and $Y(t)$. First we have to introduce the notion of a cone around a subspace.

Definition 6.2.12 *Let $\omega > 0$ and let V be a nontrivial subspace of $\mathbb{R}^d$. Then the* cone *$K_\omega(V)$ around V with angle ω is defined by*

$$K_\omega(V) := \left\{ v + w \in \mathbb{R}^d : \begin{array}{c} v \in V, w \in \mathbb{R}^d \text{ with } \langle v', w\rangle = 0 \text{ for all } v' \in V \\ \text{and } \|w\| \leq \omega \|v\| \end{array} \right\}.$$

We need the following technical lemma, which shows, that cones around X and Y can be transformed into a subset of cones around $X(t)$ and $Y(t)$.

Lemma 6.2.13 *For every $\omega > 0$ there exists a $\kappa > 0$ such that for all $t \in \mathbb{R}$*

$$\mathcal{F}^{-1}(t)\{(x,y) \in X \times Y : \ \|y\| \leq \kappa \|x\|\} \subset K_\omega(X(t))$$
$$\mathcal{F}^{-1}(t)\{(x,y) \in X \times Y : \ \|x\| \leq \kappa \|y\|\} \subset K_\omega(Y(t)).$$

Proof. We show the first relation, the second follows in the same way. Assume, that there is no such $\kappa > 0$, which means, that there is a sequence $(t_n, x_n, y_n)_{n\in\mathbb{N}} \subset [0, 2\Theta]\times X\times Y$ such that $\|y_n\| \leq \frac{1}{n}\|x_n\|$ and $\mathcal{F}^{-1}(t_n)(x,y) \notin K_\omega(X(t_n))$. We can assume, that $\|x_n\| = 1$, because if $\mathcal{F}^{-1}(t_n)(x,y) \notin K_\omega(X(t_n))$, then $\alpha\mathcal{F}^{-1}(t_n)(x,y) \notin K_\omega(X(t_n))$ for all $\alpha \in \mathbb{R}\setminus\{0\}$.

Since $\lim_{n\to\infty} y_n = 0$, there is a subsequence which we denote again by $(t_n, x_n, y_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}(t_n, x_n, y_n) = (t^*, x^*, 0) \in [0, 2\Theta] \times X \times Y$. But this means, that

$$\lim_{n\to\infty} \mathcal{F}^{-1}(t_n)(x_n, y_n) = \mathcal{F}^{-1}(t^*)(x^*, 0) \in X(t^*),$$

which is a contradiction. ■

Now the last lemma shows, that by choosing $\varepsilon > 0$ small enough, the fibres $\mathcal{X}_\varepsilon^{loc}(t), \mathcal{Y}_\varepsilon^{loc}(t)$ are arbitrarily close to their linear analogon $X(t), Y(t)$.

Lemma 6.2.14 *For every $\omega > 0$ there is an $\varepsilon(\omega) > 0$ such that*

$$\mathcal{X}_\varepsilon^{loc}(t) \subset K_\omega(X(t)) \text{ and}$$
$$\mathcal{Y}_\varepsilon^{loc}(t) \subset K_\omega(Y(t))$$

for all $\varepsilon \in (0, \varepsilon(\omega)]$ and $t \in \mathbb{R}$.

Proof. Let $L \in (0, L^*]$, then by definition of $\mathcal{X}_{\varepsilon(L)}$ and $\mathcal{Y}_{\varepsilon(L)}$ we have

$$\mathcal{X}_{\varepsilon(L)} = \{(\tau, x, y) \in \mathbb{R}\times X\times Y : y = w^+_{\varepsilon(L)}(\tau, x)\}$$
$$\mathcal{Y}_{\varepsilon(L)} = \{(\tau, x, y) \in \mathbb{R}\times X\times Y : x = w^-_{\varepsilon(L)}(\tau, y)\}$$

with

$$\left\| w^{+}_{\varepsilon(L)}(\tau, x) \right\| \le \frac{K^2 L}{\delta - KL} \|x\| \text{ and}$$
$$\left\| w^{-}_{\varepsilon(L)}(\tau, y) \right\| \le \frac{K^2 L}{\delta - KL} \|y\|$$

for every $(\tau, x, y) \in \mathbb{R}\times X \times Y$. For abbreviation we write $\kappa(L) := \frac{K^2 L}{\delta - KL}$. This means, that

$$\mathcal{X}_{\varepsilon(L)} \subset \{(\tau, x, y) \in \mathbb{R}\times X \times Y : \|y\| \le \kappa(L) \|x\|\},$$
$$\mathcal{Y}_{\varepsilon(L)} \subset \{(\tau, x, y) \in \mathbb{R}\times X \times Y : \|x\| \le \kappa(L) \|y\|\}.$$

Now according to Lemma 6.2.13 for $\omega > 0$ there exists a $\kappa > 0$ such that for all $t \in \mathbb{R}$

$$\mathcal{F}^{-1}(t)\{(t, x, y) \in \mathbb{R}\times X \times Y : \|y\| \le \kappa \|x\|\} \subset K_\omega(X(t))$$
$$\mathcal{F}^{-1}(t)\{(t, x, y) \in \mathbb{R}\times X \times Y : \|x\| \le \kappa \|y\|\} \subset K_\omega(Y(t)).$$

Thus by choosing L small enough such that $\kappa(L) = \kappa$ the assertion follows with $\varepsilon := \varepsilon(L)$.
■

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Notations

$\mathbb{N}$	$:= \{1, 2, 3, \dots\}$, the postive integers
$\mathbb{N}_0$	$:= \mathbb{N} \cup \{0\}$
$\mathbb{Z}$	the integers
$\mathbb{R}$	the real numbers
$\mathbb{R}^+$	$:= \{x \in \mathbb{R} : x > 0\}$
$\mathbb{R}_0^+$	$:= \{x \in \mathbb{R} : x \geq 0\}$
$\mathbb{R}^-$	$:= \{x \in \mathbb{R} : x < 0\}$
$\mathbb{R}_0^-$	$:= \{x \in \mathbb{R} : x \leq 0\}$
$\mathbb{R}^d$	d-dimensional vector space over $\mathbb{R}$
$\mathbb{C}$	the complex numbers
$\mathbb{S}^d$	$:= \{x \in \mathbb{R}^{d+1} : \|x\| = 1\}$, the d-dimensional unit sphere
$\mathbb{P}^d$	the d-dimensional real projective space
$\mathbf{T}M$	tangential bundle of a smooth manifold M.
$\mathbf{T}_xM$	tangent space at the point $x \in M$ of a smooth manifold$M$
$d(\cdot, \cdot)$	metric
$\mathcal{L}(X, Y)$	space of linear mappings between the vector space X and Y
$\mathcal{L}(X)$	$:= \mathcal{L}(X, X)$
$gl(X, Y)$	space of linear and invertible mappings between the vector space X and Y
$gl(X)$	$:= gl(X, X)$
C	space of continuous functions
C^n	space of n-times continuous differentiable functions
$\operatorname{cl} M$	closure of a topological space M
$\operatorname{int} M$	interior of a topological space M
$B_\varepsilon(x)$	$:= \{y \in \mathbb{R}^d : \|y\| < \varepsilon\}$ for a $x \in \mathbb{R}^d$
$B_\varepsilon(M)$	$:= \bigcup_{x \in M} B_\varepsilon(x)$ for a set $M \subset \mathbb{R}^d$
$\operatorname{diam} U$	$:= \sup\{\|x - y\| : x, y \in U\}$ diameter of $U \subset \mathbb{R}^d$
$\mathcal{U}$	admissible control functions
$\varphi(t, \tau, x, u)$	solution at time t with init value x, starting time τ and control function u
$\mathcal{O}^+_{\leq T}, \mathcal{O}^-_{\leq T}$	positive and negative orbit up to time T, page 8
$\mathcal{O}^{\mp}, \mathcal{O}^-$	positive and negative orbit, page 8
$\mathcal{LA}$	Lie algebra generated by vector fields, page 9
$\Delta_{\mathcal{LA}}(x)$	subspace of tangent space in x, generated by vector fields

	in $\mathcal{LA}$, page 9
$\mathbf{A}(D)$	domain of attraction of a control set D, page 10
$\preceq$	reachability order, page 10
θ	shift on $\mathcal{U}$, page 11
L_1	space of integrable functions
L_∞	space of locally integrable functions
Φ	control flow, page 12
$\omega(u,x)$	ω-limit set of $(u,x) \in \mathcal{U} \times M$, page 12
$[X,Y]$	Lie bracket of two vector field X, Y, page 129
$\mathrm{ad}_X Y$	$:= [X,Y]$
$\mathrm{ad}_X^k Y$	$:= \mathrm{ad}_X(\mathrm{ad}_X^{k-1} Y)$
$\mathrm{span}\{f_0, \dots, f_m\}$	vector space with basis $f_0, \dots, f_m$
$\pi_M(u,x)$	$:= x$ for $(u,x) \in \mathcal{U} \times M$
$\eta(t,\tau,u)$	fundamental solution of bilinear control system, page 16
$\mathbb{P}\eta(t,\tau,u)p$	solution of the projected control system, page 16
$\mathcal{G}$	systems group, page 17
$\mathcal{S}$	systems semigroup, page 17
$\mathrm{spec}(g)$	spectrum of a matrix g
$E(\sigma)$	generalized real eigenspace of σ
$\mathbb{P}E(\sigma)$	$:= \{\mathbb{P}x : x \in E(\sigma), x \neq 0\}$
$\prec$	reachability order between main control sets, page 18
$D(u_g)$	$:= \{x \in \mathbb{P}^{d-1}: \mathbb{P}\eta(t,0,u_g) \in \mathrm{cl}\, D$ for all $t \in \mathbb{R}\}$
$\mathbf{T}\Phi$	control flow for the bilinear control system, page 19
$\pi : \mathcal{U} \times \mathbb{R}^d \to \mathbb{R}^d$	vector bundle over $\mathbb{R}^d$
$\mathbb{P}\pi : \mathcal{U} \times \mathbb{P}^{d-1} \to \mathbb{P}^{d-1}$	projective bundle
$\mathcal{V}_1 \oplus \mathcal{V}_2$	Whitneysum of two subbundles
$\lambda(u,x)$	Lyapunov exponent, page
$\Sigma_{Fl}(D)$	Floquet spectrum over main control set D, page 21
Σ_{Fl}	Floquet spectrum of the bilinear system, page 21
$\Sigma_{Ly}(D)$	Lyapunov spectrum over main control set D, page 21
Σ_{Ly}	Lyapunov spectrum of bilinear control system, page 21
$\lambda(\zeta)$	finite time exponential growth rate of (ε,T)-chain ζ, page 21
$\Sigma_{Mo}(E)$	Morse spectrum over a chain control set E, page 21
$K_h(\mathcal{V})$	cone around a subbundle $\mathcal{V} \subset \mathcal{U} \times \mathbb{R}^d$ with angle h
Σ_{dich}	dichotomy spectrum of the linearized system, page 36
λ_i^h	Lyapunov exponents corresponding to u^h, page 48
λ_i^s	Lyapunov exponents corresponding to u^s, page 48
$\mathcal{F}$	Floquet transformation, page 167
$\psi(t,\tau,p,u^h)$	solution of the transformed system, page 52 and 167
$\mu_\varepsilon(t,\tau,p,u^h)$	solution of the restricted system, page 52 and 168
$\varepsilon(\cdot)$	maps to Lipschitz constants, page 52 and 169
L^*	maximal Lipschitz constant, for which fibre bundles exist, page 52 and 171

ε^*	maximal ε^* for which fibre bundles exist for restricted system, page 52 and 169
$\mathcal{X}_\varepsilon, \mathcal{Y}_\varepsilon$	unstable and stable fibre bundles for the restricted system, page 171
$\mathcal{X}_\varepsilon^{loc}, \mathcal{Y}_\varepsilon^{loc}$	local unstable and stable fibre bundles for the original system, page 171
$\hat{\mathcal{P}}_\varepsilon$	asymptotic phase of the restricted system, page 53
$\mathcal{P}_\varepsilon$	component in state space of asymptotic phase of the restricted system, page 53
$\hat{\mathcal{P}}_\varepsilon^{loc}$	asymptotic phase of the original system, page 53
$\mathcal{P}_\varepsilon^{loc}$	component in state space of the original system, page 53
$\mathcal{H}_\varepsilon$	topological equivalence by Hartman-Grobman of restricted system, page 53
$\mathcal{X}_{\varepsilon,\leq\rho}^{loc}$	$:= \{(t,x,y) \in \mathcal{X}_\varepsilon : \\|(x,y)\\| \leq \rho\}$
T_ρ	$:= \{(x,0) \in X \times Y : \\|x\\| = \rho\}$
D_ρ	$:= \{(x,0) \in X \times Y : \\|x\\| \leq \rho\}$
$\mathcal{T}_{\varepsilon,\rho}, \mathcal{D}_{\varepsilon,\rho}$	target and target disc for the restricted system, page 55
$\mathcal{T}_{\varepsilon,\rho}^{loc}, \mathcal{D}_{\varepsilon,\rho}^{loc}$	target and target disc for the original system, page 57
$W(\varepsilon)$	neighborhood, where restricted and original system coincide, page 57 and 172
$\hat{\varepsilon}$	maximal ε for which restricted and original system coincide locally, page 57 and 172
$\rho^*(\varepsilon)$	maximal ρ such that $\mathcal{D}_{\varepsilon,\rho}^{loc}(t) \subset W(\varepsilon)$ for all $\rho' \in (0, \rho^*(\varepsilon)$, page 57
V^s, W^s	neighborhoods where u^s-system is locally stable, page 64
$\hat{\rho} = \hat{\rho}(\varepsilon, W^s)$	$\in (0, \rho^*(\varepsilon))$ such that $\mathcal{D}_{\varepsilon,\rho}^{loc}(t) \subset W^s$ for all $\rho \in (0, \hat{\rho})$, page 64
$\mathcal{X}_{\varepsilon,>}, \mathcal{X}_{\varepsilon,<}$, $\mathcal{X}_{\varepsilon,\geq}, \mathcal{X}_{\varepsilon,\geq}$	upper and lower subspaces for the restricted system, page 78
$\mathcal{X}_{\varepsilon,>}^{loc}, \mathcal{X}_{\varepsilon,<}^{loc}$, $\mathcal{X}_{\varepsilon,\geq}, \mathcal{X}_{\varepsilon,\geq}$	upper and lower subspaces for the original system, page 78
$p_{\varepsilon,\rho,<}(t), p_{\varepsilon,\rho,>}(t)$	$:= \mathcal{X}_{\varepsilon,<}^{loc}(t) \cap \mathcal{T}_{\varepsilon,\rho}^{loc}, \mathcal{X}_{\varepsilon,>}^{loc}(t) \cap \mathcal{T}_{\varepsilon,\rho}^{loc}$
$D_<, D_>$	control sets in the onedimensional case
$\mathbf{K}$	$:= \{\alpha p : p \in \operatorname{int} D, \alpha > 0\}$ for a main control set D, page 115
$K_\omega(X)$	cone with angle ω around X
$\Sigma(A)$	dichotomy spectrum of $A \in \mathcal{L}(\mathbb{R}^d)$, page 147
$\operatorname{ess\,sup}_{t\in I} \\|f(t)\\|$	$:= \inf\{M > 0 : \\|f(t)\\| \leq M$ for a.a. $t \in I\}$, essential supremum
$\operatorname{diag}(A_1, \ldots, A_n)$	matrix with main diagonal blocks $A_1, \ldots, A_n$
E_i	vector space according to blockdiagonality of reduced standard system, page 150
$\mathcal{W}_{ij}$	integral manifold, page 156
$\mathcal{W}_{ij}[\tau^*, x^*]$	integral manifold through solution $\varphi(\cdot, \tau^*, x^*)$, page 157
$\hat{\mathcal{P}}_{1j}, \hat{\mathcal{P}}_{j+1,n}$	asymptotic phases, page 161

$\mathcal{H}$	topological equivalence by the Hartman-Grobman Theorem, page 163

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